On the structure of the multivariable free response

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***Abstract*— The structure of the free (or zero-input) response of multivariable (MIMO) linear time-invariant systems is investigated. In a behavioral setting, the free response is an autonomous behavior, solution of a homogeneous differential equation. A new closed-form expression of this solution is presented. It is a linear (real) combination of modes associated to the system's pole minimal polynomial. The vector coefficients of the modes belong to the output mode subspaces. These are characterized by a chain of subspace inclusions for each distinct pole. In the special, but relevant case of the pole minimal polynomial having simple roots the closed-form expression simplifies and admits a phasor interpretation. Examples are included to highlight the paper's findings.**

I. INTRODUCTION

The analysis of the *free* (or *zero-input*) *response* of linear time-invariant systems has a significant role in systems theory and control systems engineering [1], [2], [3], [4]. For the case of scalar systems, i.e. single-input single-output (SISO) systems, the free response is simply given by a linear combination of the (pole) modes. When considering a multivariable system, i.e. a multi-input multi-output (MIMO) system, the free response still exhibits linear combinations of modes on the output components but its actual structure non-trivially depends on the system eigenstructure (cf. [5]).

The aim of this work is to investigate the structure of the free response of multivariable linear time-invariant systems in a behavioral framework [6]. In this framework, the free response is an *autonomous behavior*, solution of a homogeneous differential equation (cf. (6)). The new main result is the closed-form expression of the free response presented in Theorem 2 (see (44) in Section V). This result appears to improve over the closed-form expression reported in [6] (Theorem 3.2.16 at page 77). Indeed, differently from [6] the expression (44) has the features:

- Despite the possible presence of complex poles all the modes and vector coefficients appearing in (44) are real. Hence, in using (44), there is no need to take real parts of complex terms.
- The modes appearing in (44) have multiplicities equal to the root multiplicities of the system's *pole minimal polynomial* and not of the *pole polynomial* (cf. Definitions 1, 2 and 3). This implies that only "true" modes appear in (44) and not modes whose vector coefficients are structural zeros.

The structure of the free response in (44) naturally leads to the new concept of *output mode subspaces* (cf. Definition 4). Indeed, a single mode can only appear on the system output on the corresponding output mode subspace (Corollary 1). Remarkably, the output mode subspaces associated to distinct system poles are related one to the other by chains of inclusions (Lemma 5). Lower bounds on the dimensions of the first and last subspace of the chain are also established (Lemma 6) by virtue of a refined result on the system observability (Proposition 1). In the special case of the pole minimal polynomial having all simple roots, the free response has a simplified closed-form expression that admits a *phasor* interpretation (Proposition 2 and Remark 4).

Paper's organization: The formulation of the addressed problem is presented in Section II. The solution approach which is based on a state-space representation of the input-output behavior is reported in Section III. Section IV reports the observability conditions in terms of the real Jordan form (Proposition 1). The findings on the multivariable free response are reported in Section V. Two illustrative examples are presented in Section VI. A brief conclusion ends the paper in Section VII. For brevity the majority of the proofs are omitted.

A. Notation

Scalars and real-valued functions are denoted by lower-case letters. Vectors and vector-valued function are denoted by bold lower-case letters such as e.g. $\boldsymbol{a} \in \mathbb{R}^n$ and matrices are denoted by capital letters such as e.g. $A \in \mathbb{R}^{n\times n}$. Given a matrix $B \in \mathbb{R}^{n\times m}$, its entries are denoted by b_{ij}, $i = 1,\dots,n$, $j = 1,\dots,m$ so that $B = [b_{ij}]$. The image of B is $\operatorname{im} B := \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{x} = B\boldsymbol{v},\ \boldsymbol{v} \in \mathbb{R}^m\}$.

The set of polynomials with real coefficients in the indeterminate s is denoted by $\mathbb{R}[s]$ and if $g \in \mathbb{R}[s]$ then $\deg g$ denotes its degree. A $p \times m$ polynomial matrix with entries in $\mathbb{R}[s]$ is said to belong to $\mathbb{R}[s]^{p\times m}$.

A function $\boldsymbol{f} = [f_1,\dots,f_n]^T : \mathbb{R} \to \mathbb{R}^n$ belongs to $C^\infty(\mathbb{R},\mathbb{R}^n)$ if for any component of $\boldsymbol{f}$ there exist derivatives of any order. The derivative operator is denoted by D. Given a real function $f(t)$, the integral operator is defined by $\int f(t) := \int_0^t f(v)dv$ and $\int^0 f := f$. Let $n \in \mathbb{N}$ with $n \geq 1$, $\int^n f$ is defined by recursion: $\int^n f := \int(\int^{n-1} f)$.

II. ADDRESSED PROBLEM

Consider a linear time-invariant system H defined by its matrix transfer function

$$H(s) = [h_{ij}(s)],\ i = 1,\dots,p;\ j = 1,\dots,m. \tag{1}$$

$H(s)$ has full rank and its entries $h_{ij}(s)$ are strictly proper rational functions given by the ratio of coprime polynomials. The input and output of H are $\boldsymbol{u} \in C^\infty_{\mathrm{p}}(\mathbb{R},\mathbb{R}^m)$ and $\boldsymbol{y} \in$

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$C^\infty_{\mathtt{p}}(\mathbb{R},\mathbb{R}^p)$, the sets of piecewise C^∞-functions from $\mathbb{R}$ to $\mathbb{R}^m$ and to $\mathbb{R}^p$ respectively [7], [8]. The transfer function (1) can be written as a left-coprime matrix fraction description (MFD)

$$H(s) = P(s)^{-1}Q(s) \tag{2}$$

in which $P(s) \in \mathbb{R}[s]^{p\times p}$ and $Q(s) \in \mathbb{R}[s]^{p\times m}$ are polynomial matrices [9]. The MFD (2) naturally introduces the *behavior* of H, $\mathcal{B}_H$ as the set of weak solutions of the differential equation $P(D)\boldsymbol{y}(t) = Q(D)\boldsymbol{u}(t)$, $t\in\mathbb{R}$ [6], [7], [8]:

$$\mathcal{B}_H := \big\{(\boldsymbol{u},\boldsymbol{y}) \in C^\infty_{\mathtt{p}}(\mathbb{R},\mathbb{R}^m)\times C^\infty_{\mathtt{p}}(\mathbb{R},\mathbb{R}^p) : P(D)\boldsymbol{y}(t) = Q(D)\boldsymbol{u}(t) \text{ weakly}\big\}\,. \tag{3}$$

Specifically, if $P(s) = \sum_{i=0}^{n_P} P_i s^i$ and $Q(s) = \sum_{i=0}^{n_Q} Q_i s^i$ a pair $(\boldsymbol{u},\boldsymbol{y})$ belongs to $\mathcal{B}_H$ if there exists $\boldsymbol{g}(t) \in \mathbb{R}^p[t]$, $\deg \boldsymbol{g} \le n_P - 1$ such that the following integral equation is satisfied [7], [8]

$$\sum_{i=0}^{n_P} P_i \int^{n_P-i} \boldsymbol{y}(t) = \sum_{i=0}^{n_Q} Q_i \int^{n_P-i} \boldsymbol{u}(t) + \boldsymbol{g}(t),\ t\in\mathbb{R}. \tag{4}$$

When the input is identically zero, i.e. $u \equiv 0$, correspondingly the output exhibits the so-called *(multivariable) free response*. This can be found as a solution of a homogeneous differential equation in the following subset of $\mathcal{B}_H$:

$$\mathcal{B}_{H,y\text{-hom}} := \{(0,\boldsymbol{y}_{\mathtt{hom}}) : P(D)\boldsymbol{y}_{\mathtt{hom}}(t) = 0 \text{ weakly}\}\,. \tag{5}$$

In the set definition (5) we can drop the specification "weakly" due to the following result (cf. [6], [7]).

***Lemma** 1:* Let $P(s) \in \mathbb{R}^{p\times p}[s]$ with $\det P(s) \neq 0$. A function $\boldsymbol{y}_{\mathtt{hom}} \in C^\infty_{\mathtt{p}}(\mathbb{R},\mathbb{R}^p)$ is a weak solution of $P(D)\boldsymbol{y}_{\mathtt{hom}}(t) = 0$ if and only if it is a (strong) solution belonging to $C^\infty(\mathbb{R},\mathbb{R}^p)$.

The addressed problem is to exploit the structure of $\boldsymbol{y}_{\mathtt{hom}}(t)$, i.e. the structure of the solution of the homogeneous differential equation

$$P(D)\boldsymbol{y}_{\mathtt{hom}}(t) = 0,\ t\in\mathbb{R}\,. \tag{6}$$

Relevant to this problem are the following definitions. Poles of H can be introduced as the roots of the *pole polynomial*.

***Definition** 1 ([10]):* The *pole polynomial* of H, $p_H(s)$ is the monic least common denominator of all nonzero minors of $H(s)$ (cf. (1)).

***Definition** 2 ([10]):* The *pole minimal polynomial* of H, $p'_H(s)$ is the monic least common denominator of all nonzero entries of $H(s)$ (cf. (1)).

As known $p'_H(s)$ divides $p_H(s)$ and the sets of the distinct roots of $p'_H(s)$ and $p_H(s)$ coincide. The *pole* (or *system*) *modes* of H are associated to the pole minimal polynomial $p'_H(s)$ according to this definition (cf. [5]).

***Definition** 3 (pole modes):* Given a real pole $\rho \in \mathbb{R}$ (complex pole $\sigma \pm j\omega \in \mathbb{C}$) with multiplicity μ (ν) as a root of the pole minimal polynomial p'_H, the associated *pole modes* are the time-functions $e^{\rho t}, te^{\rho t}, \ldots, t^{\mu-1}e^{\rho t}$ ($e^{\sigma t}\cos(\omega t), e^{\sigma t}\sin(\omega t), \ldots, t^{\nu-1}e^{\sigma t}\cos(\omega t), t^{\nu-1}e^{\sigma t}\sin(\omega t)$).

III. Solution Approach

Introduce a minimal state-space realization of $H(s)$:

$$\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + B\boldsymbol{u}(t)\,, \tag{7}$$

$$\boldsymbol{y}(t) = C\boldsymbol{x}(t)\,. \tag{8}$$

with $A\in\mathbb{R}^{n\times n}$, $B\in\mathbb{R}^{n\times m}$ and $C\in\mathbb{R}^{p\times n}$ for which $H(s) = C(sI-A)^{-1}B$ and (A,B) is controllable and (C,A) is observable [9]. Hence, the characteristic polynomial of A, i.e. $\det(sI-A)$ coincides with the pole polynomial $p_H(s)$ and the minimal polynomial of A coincides with the pole minimal polynomial $p'_H(s)$ [10], [5].

A weak solution of the differential equation (7) is a time function $\boldsymbol{x}\in C^\infty_{\mathtt{p}}(\mathbb{R},\mathbb{R}^n)$ for which there exists a constant $\boldsymbol{c}\in\mathbb{R}^n$ such that

$$\boldsymbol{x}(t) = A\int \boldsymbol{x}(t) + B\int \boldsymbol{u} + \boldsymbol{c}\,, t\in\mathbb{R}\,. \tag{9}$$

Hence, the behavior of H can be equivalently represented as follows [8]:

$$\mathcal{B}_H = \big\{(\boldsymbol{u},\boldsymbol{y}) \in C^\infty_{\mathtt{p}}(\mathbb{R},\mathbb{R}^m)\times C^\infty_{\mathtt{p}}(\mathbb{R},\mathbb{R}^p) : \exists\boldsymbol{x}\in C^\infty_{\mathtt{p}}(\mathbb{R},\mathbb{R}^n) \text{ with } \boldsymbol{y}(t) = C\boldsymbol{x}(t),\ t\in\mathbb{R} \text{ and } \dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + B\boldsymbol{u}(t) \text{ weakly}\big\}\,. \tag{10}$$

By setting $\boldsymbol{u}\equiv 0$ in the set definition (10) we have

$$\mathcal{B}_{H,y\text{-hom}} = \big\{(0,\boldsymbol{y}_{\mathtt{hom}}) : \exists\boldsymbol{x}\in C^\infty_{\mathtt{p}}(\mathbb{R},\mathbb{R}^n) \text{ with } \boldsymbol{y}_{\mathtt{hom}}(t) = C\boldsymbol{x}(t),\ t\in\mathbb{R} \text{ and } \dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) \text{ weakly}\big\}\,. \tag{11}$$

Since $\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t)$ is a linear homogeneous differential equation we can drop "weakly" in (11) by virtue of Lemma 1 (as analogously done in (5)). As known, the solution of $\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t)$ is given by the matrix exponential $e^{At}\boldsymbol{x}(0)$ in which $\boldsymbol{x}(0)$ is the state of model (7)-(8) at time zero. Hence, the study of the structure of the free response $\boldsymbol{y}_{\mathtt{hom}}(t)$ (cf. (6)) can be based on the following expression:

$$\boldsymbol{y}_{\mathtt{hom}}(t) = Ce^{At}\boldsymbol{x}(0),\ t\in\mathbb{R}. \tag{12}$$

IV. Observability Conditions

In studying the structure of $\boldsymbol{y}_{\mathtt{hom}}(t)$ as expressed by (12), it is important to take into account the observability property of pair (C,A). As known, for any similarity transformation T pair (C,A) is observable if and only if pair $(CT, T^{-1}AT)$ is observable [10]. Hence, the classic observability conditions of Chen and Desoer [11] can be recalled. This requires to transform the system matrix A in its Jordan form. Let $J := T^{-1}AT$ and $E := CT$ and denote by $\lambda_1,\ldots,\lambda_d$ the distinct eigenvalues of A (i.e. the distinct poles of H). Matrices $J\in\mathbb{C}^{n\times n}$ and $E\in\mathbb{C}^{p\times n}$ are partitioned as follows (both J and E have complex entries when complex eigenvalues occur):

$$J = \mathrm{diag}\{J_1,\ldots,J_d\},$$
$$E = [E_1 \cdots E_d]$$

and with $i = 1, \dots, d$

$$J_i = \text{diag}\{J_{i1}, \dots, J_{ib(i)}\} \;\; (n_i \times n_i),$$
$$E_i = [E_{i1} \cdots E_{ib(i)}] \;\; (p \times n_i)$$

in which $b(i)$ denotes the number of Jordan blocks in J_i ($j = 1, \dots, b(i)$):

$$J_{ij} = \begin{bmatrix} \lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{bmatrix} \;\; (n_{ij} \times n_{ij}),$$
$$E_{ij} = [\boldsymbol{e}_{ij1} \cdots\cdots \boldsymbol{e}_{ijn_{ij}}] \;\; (p \times n_{ij}).$$

***Theorem** 1 (Chen, Desoer [11]):* (C, A) is observable if and only if, for each $i = 1, \dots, d$, the $b(i)$ p-dimensional vectors

$$\boldsymbol{e}_{i11}, \boldsymbol{e}_{i21}, \dots, \boldsymbol{e}_{ib(i)1} \tag{13}$$

are linearly independent.

To our aim we also introduce the real Jordan form of A [12]. Among the λ_i eigenvalues we distinguish between the distinct real eigenvalues $\rho_1, \dots, \rho_l$ and the distinct complex ones $\sigma_1 \pm j\omega_1, \dots, \sigma_k \pm j\omega_k$. There exists a real similarity transformation $T_{\mathtt{r}}$ leading to $J_{\mathtt{r}} = T_{\mathtt{r}}^{-1} A T_{\mathtt{r}} \in \mathbb{R}^{n \times n}$, $[KL] = CT_{\mathtt{r}} \in \mathbb{R}^{p \times n}$ partitioned as follows

$$J_{\mathtt{r}} = \text{diag}\{R_1, \dots, R_l, C_1, \dots, C_k\}, \tag{14}$$
$$[KL] = [K_1 \cdots K_l L_1 \cdots L_k]. \tag{15}$$

Matrices R_i and K_i, $i = 1, \dots, l$ are structured as

$$R_i = \text{diag}\{R_{i1}, \dots, R_{ir(i)}\} \;\; (l_i \times l_i), \tag{16}$$
$$K_i = [K_{i1} \cdots K_{ir(i)}] \;\; (p \times l_i) \tag{17}$$

in which $r(i)$ denotes the number of Jordan blocks associated to ρ_i and with $j = 1, \dots, r(i)$

$$R_{ij} = \begin{bmatrix} \rho_i & 1 & & \\ & \rho_i & \ddots & \\ & & \ddots & 1 \\ & & & \rho_i \end{bmatrix} \;\; (l_{ij} \times l_{ij}), \tag{18}$$
$$K_{ij} = [\boldsymbol{k}_{ij1} \cdots\cdots \boldsymbol{k}_{ijl_{ij}}] \;\; (p \times l_{ij}). \tag{19}$$

Matrices C_i and L_i, $i = 1, \dots, k$ are structured as

$$C_i = \text{diag}\{C_{i1}, \dots, C_{ic(i)}\} \;\; (2k_i \times 2k_i), \tag{20}$$
$$L_i = [L_{i1} \cdots L_{ic(i)}] \;\; (p \times 2k_i) \tag{21}$$

in which $c(i)$ denotes the number of real Jordan blocks associated to $\sigma_i \pm j\omega_i$ and with $j = 1, \dots, c(i)$

$$C_{ij} = \begin{bmatrix} D_i & I_2 & & \\ & D_i & \ddots & \\ & & \ddots & I_2 \\ & & & D_i \end{bmatrix} \;\; (2k_{ij} \times 2k_{ij}), \tag{22}$$
$$D_i = \begin{bmatrix} \sigma_i & \omega_i \\ -\omega_i & \sigma_i \end{bmatrix}, \;\; I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{23}$$
$$L_{ij} = [\boldsymbol{m}_{ij1} \boldsymbol{n}_{ij1} \cdots\cdots \boldsymbol{m}_{ijk_{ij}} \boldsymbol{n}_{ijk_{ij}}] \;\; (p \times 2k_{ij}). \tag{24}$$

***Remark** 1:* In relation to the introduced real Jordan form the pole polynomial p_H is factorized as follows

$$p_H(s) = \prod_{i=1}^{l} (s - \rho_i)^{l_i} \prod_{i=1}^{k} \left[(s - \sigma_i)^2 + \omega_i^2\right]^{k_i}. \tag{25}$$

Without loss of generality the following orderings of Jordan block dimensions are introduced

$$\mu_i := l_{i1} \geq l_{i2} \geq \cdots \geq l_{ir(i)}, \; i = 1, \dots, l \tag{26}$$
$$\nu_i := k_{i1} \geq k_{i2} \geq \cdots \geq k_{ic(i)}, \; i = 1, \dots, k \tag{27}$$

Hence, the pole minimal polynomial p'_H can be expressed as

$$p'_H(s) = \prod_{i=1}^{l} (s - \rho_i)^{\mu_i} \prod_{i=1}^{k} \left[(s - \sigma_i)^2 + \omega_i^2\right]^{\nu_i}. \tag{28}$$

A useful technical result is the following.

***Lemma** 2:* The complex vectors $\boldsymbol{a}_i + j\boldsymbol{b}_i \in \mathbb{C}^n$, $i = 1, \dots, k$ are linearly independent if and only if the conjugate vectors $\boldsymbol{a}_i - j\boldsymbol{b}_i \in \mathbb{C}^n$, $i = 1, \dots, k$ are linearly independent.

With reference to the introduced real Jordan form the observability conditions can be stated as follows.

***Proposition** 1:* (C, A) is observable if and only if

$$\text{rank}[\boldsymbol{k}_{i11} \boldsymbol{k}_{i21} \cdots \boldsymbol{k}_{ir(i)1}] = r(i), \; i = 1, \dots, l \tag{29}$$

and

$$\text{rank}[\boldsymbol{m}_{i11} + j\boldsymbol{n}_{i11}, \boldsymbol{m}_{i21} + j\boldsymbol{n}_{i21}, \cdots, \boldsymbol{m}_{ic(i)1} + j\boldsymbol{n}_{ic(i)1}] = c(i), \; i = 1, \dots, k. \tag{30}$$

***Remark** 2:* The Chen and Desoer's Theorem 1 is a compact and elegant statement. However, it contains redundant conditions when complex eigenvalues occur. Indeed there are $d = l + 2k$ conditions in Theorem 1 whereas Proposition 1 contains just $l + k$ conditions. The k redundant conditions of Theorem 1 are conjugate conditions that can be omitted (cf. Lemma 2).

V. Structure of the Multivariable Free Response

By using the real similarity transformation $T_{\mathtt{r}}$ (cf. Section IV) the homogeneous solution $\boldsymbol{y}_{\mathtt{hom}}(t)$ in (12) can be expressed as

$$\boldsymbol{y}_{\mathtt{hom}}(t) = [KL] e^{J_{\mathtt{r}} t} z(0), \; t \in \mathbb{R} \tag{31}$$

with $z(0) := T_{\mathtt{r}}^{-1} x(0)$. The real Jordan form $J_{\mathtt{r}}$ is a block diagonal matrix defined in (14) for which the corresponding matrix exponential is still block diagonal:

$$e^{J_{\mathtt{r}} t} = \text{diag}\left\{e^{R_1 t}, \dots, e^{R_l t}, e^{C_1 t}, \dots, e^{C_k t}\right\}.$$

By taking into account the partitioning of matrices K and L in (15) and by introducing a congruent partitioning of $z(0)$ as

$$[\boldsymbol{r}_1^T \cdots \boldsymbol{r}_l^T \boldsymbol{c}_1^T \cdots \boldsymbol{c}_k^T]^T := z(0) \tag{32}$$

we obtain

$$\boldsymbol{y}_{\mathtt{hom}}(t) = \sum_{i=1}^{l} K_i e^{R_i t} \boldsymbol{r}_i + \sum_{i=1}^{k} L_i e^{C_i t} \boldsymbol{c}_i, \; t \in \mathbb{R} \tag{33}$$

with $\boldsymbol{r}_i \in \mathbb{R}^{l_i}$ and $\boldsymbol{c}_i \in \mathbb{R}^{2k_i}$.

The matrix exponentials $e^{R_i t}$ and $e^{C_i t}$ contains the modes associated to ρ_i and $\sigma_i \pm j\omega_i$ respectively (cf. Definition 3). How these modes appear on the system output is explained by the lemmas below.

***Lemma** 3:* Let $i = 1, \dots, l$ and $j = 1, \dots, \mu_i$ and

$$\Gamma_{ij} := \text{diag}\left\{\Pi_{\mu_i j}, \Pi_{l_{i2} j}, \dots, \Pi_{l_{ir(i)} j}\right\} \tag{34}$$

with $\Pi_{sj} \in \mathbb{R}^{s \times s}$, $s = 1, \dots, \mu_i$ and $\Pi_{sj} := 0$ if $j > s$,

$$\Pi_{sj} := \begin{bmatrix} 0 & \overset{\downarrow j\text{-th column}}{\frac{1}{(j-1)!}} & & 0 \\ & \ddots & & \ddots & \\ \vdots & & \ddots & & \frac{1}{(j-1)!} \\ & & & \ddots & \\ 0 & & \cdots & & 0 \end{bmatrix} \text{ if } j \leq s\,. \tag{35}$$

Then

$$K_i e^{R_i t} \boldsymbol{r}_i = \sum_{j=1}^{\mu_i} K_i \Gamma_{ij} \boldsymbol{r}_i t^{j-1} e^{\rho_i t}\,,\ t \in \mathbb{R}. \tag{36}$$

Proof: The $l_i \times l_i$ matrix R_i is block diagonal (cf. (16)) so that

$$K_i e^{R_i t} \boldsymbol{r}_i = K_i \begin{bmatrix} e^{R_{i1} t} & & \\ & \ddots & \\ & & e^{R_{ir(i)} t} \end{bmatrix} \boldsymbol{r}_i. \tag{37}$$

The matrix exponentials in (37) associated to the Jordan blocks R_{ik}, $k = 1, \dots, r(i)$ (cf. (18)) are upper triangular Toeplitz matrices that contains the modes $e^{\rho_i t}$, $te^{\rho_i t}$, $\dots$, $t^{l_{ik}-1} e^{\rho_i t}$ [13]

$$e^{R_{ik} t} = \begin{bmatrix} e^{\rho_i t} & te^{\rho_i t} & & \frac{t^{l_{ik}-1}}{(l_{ik}-1)!} e^{\rho_i t} \\ 0 & \ddots & \ddots & \\ & \ddots & \ddots & te^{\rho_i t} \\ 0 & & 0 & e^{\rho_i t} \end{bmatrix}. \tag{38}$$

Hence, the mode $t^{j-1} e^{\rho_i t}$, $j = 1, \dots, \mu_i$ appears in $e^{R_i t}$ by the contribution of the matrix exponentials (38) having $l_{ik} \geq j$ ($e^{R_{ik} t}$ is a $l_{ik} \times l_{ik}$ matrix). Then, the matrix Γ_{ij} selects the columns of K_i associated to $\frac{1}{(j-1)!} t^{j-1} e^{\rho_i t}$ in such a way that the resulting vector coefficient of $t^{j-1} e^{\rho_i t}$ is $K_i \Gamma_{ij} \boldsymbol{r}_i$. ■

***Lemma** 4:* Let $i = 1, \dots, k$ and $j = 1, \dots, \nu_i$ and

$$\Psi_{ij} = \text{diag}\left\{\Omega_{\nu_i j}, \Omega_{k_{i2} j}, \dots, \Omega_{k_{ic(i)} j}\right\}\,, \tag{39}$$

$$\widetilde{\Psi}_{ij} = \text{diag}\left\{\widetilde{\Omega}_{\nu_i j}, \widetilde{\Omega}_{k_{i2} j}, \dots, \widetilde{\Omega}_{k_{ic(i)} j}\right\} \tag{40}$$

with $\Omega_{sj}, \widetilde{\Omega}_{sj} \in \mathbb{R}^{2s \times 2s}$, $s = 1, \dots, \nu_i$ and $\Omega_{sj} = \widetilde{\Omega}_{sj} := 0$ if $j > s$,

$$\Omega_{sj} := \begin{bmatrix} O_2 & \overset{\downarrow j\text{-th block column}}{\Delta_j} & O_2 \\ & \ddots & & \ddots & \\ \vdots & & \ddots & & \Delta_j \\ & & & \ddots & \\ O_2 & & \cdots & & O_2 \end{bmatrix} \text{ if } j \leq s \tag{41}$$

$$O_2 := \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\,,\ \Delta_j := \begin{bmatrix} \frac{1}{(j-1)!} & 0 \\ 0 & \frac{1}{(j-1)!} \end{bmatrix}. \tag{42}$$

$\widetilde{\Omega}_{sj}$ is defined as in (41) with the substitution of Δ_j with $\widetilde{\Delta}_j := \begin{bmatrix} 0 & \frac{1}{(j-1)!} \\ -\frac{1}{(j-1)!} & 0 \end{bmatrix}$. Then

$$L_i e^{C_i t} \boldsymbol{c}_i = \sum_{j=1}^{\nu_i} L_i \Big\{ \Psi_{ij} \boldsymbol{c}_i t^{j-1} e^{\sigma_i t} \cos(\omega_i t) + \widetilde{\Psi}_{ij} \boldsymbol{c}_i t^{j-1} e^{\sigma_i t} \sin(\omega_i t) \Big\}\,,\ t \in \mathbb{R}. \tag{43}$$

By collecting the results from Lemma 3 and 4 the general expression for the multivariable free response is obtained.

***Theorem** 2:* The set of all solutions of the homogeneous differential equation (6) is given by

$$\boldsymbol{y}_{\texttt{hom}}(t) = \sum_{i=1}^{l} \sum_{j=1}^{\mu_i} K_i \Gamma_{ij} \boldsymbol{r}_i t^{j-1} e^{\rho_i t} + \sum_{i=1}^{k} \sum_{j=1}^{\nu_i} L_i \Big\{ \Psi_{ij} \boldsymbol{c}_i t^{j-1} e^{\sigma_i t} \cos(\omega_i t) + \widetilde{\Psi}_{ij} \boldsymbol{c}_i t^{j-1} e^{\sigma_i t} \sin(\omega_i t) \Big\},\ t \in \mathbb{R} \tag{44}$$

with $\boldsymbol{r}_i \in \mathbb{R}^{l_i}$, $i = 1, \dots, l$ and $\boldsymbol{c}_i \in \mathbb{R}^{2k_i}$, $i = 1, \dots, k$.

The vector coefficients of the modes in (44) are $K_i \Gamma_{ij} \boldsymbol{r}_i$, $L_i \Psi_{ij} \boldsymbol{c}_i$ and $L_i \widetilde{\Psi}_{ij} \boldsymbol{c}_i$ with $\boldsymbol{r}_i \in \mathbb{R}^{l_i}$ and $\boldsymbol{c}_i \in \mathbb{R}^{2k_i}$. Their structure naturally leads to the introduction of the *output mode subspaces*.

***Definition** 4 (output mode subspaces):* Let $i = 1, \dots, l$ and $j = 1, \dots, \mu_i$. The output mode subspace associated to $t^{j-1} e^{\rho_i t}$ is

$$\mathcal{F}_{ij} := \text{im}\, K_i \Gamma_{ij}\,. \tag{45}$$

Let $i = 1, \dots, l$ and $j = 1, \dots, \nu_i$. The output mode subspace associated to $t^{j-1} e^{\sigma_i t} \cos(\omega_i t)$ and $t^{j-1} e^{\sigma_i t} \sin(\omega_i t)$ is

$$\mathcal{G}_{ij} := \text{im}\, L_i \Psi_{ij}\,. \tag{46}$$

***Remark** 3:* The definition of $\mathcal{G}_{ij}$ in (46) could have been given by $\text{im}\, L_i \widetilde{\Psi}_{ij}$ as well because $\text{im}\, L_i \Psi_{ij} = \text{im}\, L_i \widetilde{\Psi}_{ij}$. Indeed, permutations and sign-changes of columns that characterize $\widetilde{\Psi}_{ij}$ with respect Ψ_{ij} do not modify the resulting span of columns (cf. Lemma 4).

Significantly, the output mode subspaces associated to the distinct poles of H are related one to the other by the following chains of inclusions.

***Lemma** 5:* The output mode subspaces satisfy the following inclusions. Let $i = 1, \dots, l$ and $\mu_i > 1$ then

$$\mathcal{F}_{i1} \supseteq \mathcal{F}_{i2} \supseteq \cdots \supseteq \mathcal{F}_{i\mu_i}\,. \tag{47}$$

Let $i=1,\dots,k$ and $\nu_i>1$ then

$$\mathcal{G}_{i1}\supseteq\mathcal{G}_{i2}\supseteq\cdots\supseteq\mathcal{G}_{i\nu_i}\,. \tag{48}$$

The subset inclusions appearing in (47) and (48) may be strict or not strict depending on the actual system H (cf. Example 1 in Section VI). In any case, the observability conditions (cf. Proposition 1) dictate the following lower bounds on the dimensions of the first and last subspace in the chains (47) and (48).

***Lemma** 6:* Let $i=1,\dots,l$. Then

$$\dim\mathcal{F}_{i1}\geq r(i),\ \dim\mathcal{F}_{i\mu_i}\geq 1. \tag{49}$$

Let $i=1,\dots,k$. Then

$$\dim\mathcal{G}_{i1}\geq c(i),\ \dim\mathcal{F}_{i\nu_i}\geq 1. \tag{50}$$

Functions $r(i)$ and $c(i)$ denote the number of real Jordan blocks associated to ρ_i and $\sigma_i\pm j\omega_i$ respectively (cf. Section IV).

The introduction of the output mode subspaces allows to express the free response (44) in the following way.

***Corollary** 1:* Let $\boldsymbol{y}_{\texttt{hom}}(t)$, $t\in\mathbb{R}$ be a solution of the homogeneous differential equation (6). Then there exist output vectors $\boldsymbol{f}_{ij}\in\mathcal{F}_{ij}$, $i=1,\dots,l$, $j=1,\dots,\mu_i$ and $\boldsymbol{g}_{ij},\widetilde{\boldsymbol{g}}_{ij}\in\mathcal{G}_{ij}$, $i=1,\dots,k$, $j=1,\dots,\nu_i$ such that

$$\begin{aligned}\boldsymbol{y}_{\texttt{hom}}(t)&=\sum_{i=1}^{l}\sum_{j=1}^{\mu_i}\boldsymbol{f}_{ij}t^{j-1}e^{\rho_i t}\\&+\sum_{i=1}^{k}\sum_{j=1}^{\nu_i}\Big\{\boldsymbol{g}_{ij}t^{j-1}e^{\sigma_i t}\cos(\omega_i t)+\widetilde{\boldsymbol{g}}_{ij}t^{j-1}e^{\sigma_i t}\sin(\omega_i t)\Big\}.\end{aligned} \tag{51}$$

The elaborate structure of the general solution (44) simplifies when considering special cases. A particular significant one occurs when the roots of the pole minimal polynomial $p'_H(s)$ are all simple, i.e. $\mu_i=1$, $i=1,\dots,l$ and $\nu_i=1$, $i=1,\dots,k$ or equivalently the system matrix A is diagonalizable [13].

***Proposition** 2 (A is diagonalizable):* Let $\mu_i=1$, $i=1,\dots,l$ and $\nu_i=1$, $i=1,\dots,k$. Then the set of all solutions of the homogeneous differential equation (6) is given by

$$\begin{aligned}\boldsymbol{y}_{\texttt{hom}}(t)&=\sum_{i=1}^{l}K_i\boldsymbol{r}_i e^{\rho_i t}+\sum_{i=1}^{k}\sum_{j=1}^{k_i}\gamma_{ij}e^{\sigma_i t}\\&\cdot\Big\{\boldsymbol{m}_{ij1}\cos(\omega_i t+\delta_{ij})+\boldsymbol{n}_{ij1}\cos(\omega_i t+\delta_{ij}+\pi/2)\Big\},\ t\in\mathbb{R}\end{aligned} \tag{52}$$

with $\boldsymbol{r}_i\in\mathbb{R}^{l_i}$, $i=1,\dots,l$ and $\gamma_{ij}\geq 0$, $\delta_{ij}\in[0,2\pi)$, $i=1,\dots,k$, $j=1,\dots,k_i$.

***Remark** 4:* The cosine functions appearing in (52) have a noteworthy representation as *phasors* [14]. Indeed, the time-functions $\boldsymbol{m}_{ij1}\cos(\omega_i t+\delta_{ij})+\boldsymbol{n}_{ij1}\cos(\omega_i t+\delta_{ij}+\pi/2)$, $i=1,\dots,k$ and $j=1,\dots,k_i$ are associated to the phasors $\boldsymbol{m}_{ij1}e^{j\delta_{ij}}+j\boldsymbol{n}_{ij1}e^{j\delta_{ij}}$ and, by factoring out $e^{j\delta_{ij}}$, to $\boldsymbol{m}_{ij1}+j\boldsymbol{n}_{ij1}$. For each $i=1,\dots,k$ these phasors are linearly independent (over the field of complex numbers) by virtue of the observability conditions (30) of Proposition 1 ($c(i)=k_i$ in this case).

VI. Illustrative Examples

A. Example 1

Let us consider a system whose transfer function is

$$H(s)=\begin{bmatrix}\frac{1}{(s+1)^2}&\frac{1}{(s+1)^3}&-\frac{1}{(s+1)^2}\\\frac{2}{s+1}&\frac{1}{(s+1)^2}&0\\-\frac{1}{s+1}&\frac{2s^2+4s+1}{(s+1)^2(s+2)}&\frac{1}{s+1}\end{bmatrix}. \tag{53}$$

A coprime MFD of $H(s)$ is $P^{-1}(s)Q(s)$ with

$$P(s)=\begin{bmatrix}s^2+2s+1&-s-1&0\\0&s^2+2s+1&0\\s+1&0&s+2\end{bmatrix}, \tag{54}$$

$$Q(s)=\begin{bmatrix}-1&0&-1\\2s+2&1&0\\-1&2&1\end{bmatrix}. \tag{55}$$

The multivariable free response of H is given by the solution of the homogeneous differential equation $P(D)\boldsymbol{y}_{\texttt{hom}}(t)=0$ (cf. (6)). A preamble to exhibit the structure of this solution is to find a minimal state-space realization $\{A,B,C\}$ of $H(s)$ (cf. (7)-(8) and [9]). Then, by a suitable similarity transformation $T_{\mathtt{r}}$ the real Jordan form $J_{\mathtt{r}}=T_{\mathtt{r}}^{-1}AT_{\mathtt{r}}$ is determined along with $[KL]=CT_{\mathtt{r}}$ (cf. Section IV):

$$J_{\mathtt{r}}=\begin{bmatrix}-1&1&0&0&0\\0&-1&1&0&0\\0&0&-1&0&0\\0&0&0&-1&0\\0&0&0&0&-2\end{bmatrix}, \tag{56}$$

$$K=\begin{bmatrix}1&0&-1&0&0\\0&1&0&1&0\\0&-1&1&0&1\end{bmatrix},\ L=0. \tag{57}$$

Apply Theorem 2 to obtain ($l=2$, $\mu_1=3$, $\mu_2=1$):

$$\begin{aligned}\boldsymbol{y}_{\texttt{hom}}(t)=&K_1\Gamma_{11}\boldsymbol{r}_1e^{-t}+K_1\Gamma_{12}\boldsymbol{r}_1te^{-t}+K_1\Gamma_{13}\boldsymbol{r}_1t^2e^{-t}\\&+K_2\Gamma_{21}\boldsymbol{r}_2e^{-2t}\,.\end{aligned}$$

From (15) and (34), (35) we obtain

$$\begin{aligned}\boldsymbol{y}_{\texttt{hom}}(t)=&\begin{bmatrix}1&0&-1&0\\0&1&0&1\\0&-1&1&0\end{bmatrix}\boldsymbol{r}_1e^{-t}+\begin{bmatrix}0&1&0&0\\0&0&1&0\\0&0&-1&0\end{bmatrix}\boldsymbol{r}_1te^{-t}\\&+\begin{bmatrix}0&0&1/2&0\\0&0&0&0\\0&0&0&0\end{bmatrix}\boldsymbol{r}_1t^2e^{-t}+\begin{bmatrix}0\\0\\1\end{bmatrix}\boldsymbol{r}_2e^{-2t}\,.\end{aligned}$$

By parameterizing $\boldsymbol{r}_1:=[\alpha_1\alpha_2\alpha_3\alpha_4]^T\in\mathbb{R}^4$ and $\boldsymbol{r}_2:=\alpha_5\in\mathbb{R}$ the following explicit parameterization of the free response is obtained:

$$\begin{aligned}\boldsymbol{y}_{\texttt{hom}}(t)=&\begin{bmatrix}\alpha_1-\alpha_3\\\alpha_2+\alpha_4\\-\alpha_2+\alpha_3\end{bmatrix}e^{-t}+\begin{bmatrix}\alpha_2\\\alpha_3\\-\alpha_3\end{bmatrix}te^{-t}\\&+\begin{bmatrix}\alpha_3/2\\0\\0\end{bmatrix}t^2e^{-t}+\begin{bmatrix}0\\0\\\alpha_5\end{bmatrix}e^{-2t}\,.\end{aligned} \tag{58}$$

The system modes in (58) appear on the following output (mode) subspaces (cf. Definition 4):

$$\mathcal{F}_{11} = \operatorname{im} K_1\Gamma_{11} = \mathbb{R}^3,\ \mathcal{F}_{12} = \operatorname{im} K_1\Gamma_{12} = \operatorname{im}\begin{bmatrix}1 & 0\\ 0 & 1\\ 0 & -1\end{bmatrix}, \tag{59}$$

$$\mathcal{F}_{13} = \operatorname{im} K_1\Gamma_{13} = \operatorname{im}\begin{bmatrix}1\\0\\0\end{bmatrix},\ \mathcal{F}_{21} = \operatorname{im} K_2\Gamma_{21} = \operatorname{im}\begin{bmatrix}0\\0\\1\end{bmatrix} \tag{60}$$

associated to e^{-t}, te^{-t}, t^2e^{-t}, e^{-2t} respectively.

B. Example 2

A system has the transfer function

$$H(s) = \begin{bmatrix}\frac{s-1}{s^2-2s+5} & \frac{5s-9}{(s^2-2s+5)(s+4)}\\ \frac{2s^2+s+1}{(s^2-2s+5)(s+4)} & 0\\ \frac{2}{s^2-2s+5} & \frac{s^2+13}{(s^2-2s+5)(s+4)}\end{bmatrix} \tag{61}$$

for which a coprime MFD $H(s) = P^{-1}(s)Q(s)$ is given by

$$P(s) = \begin{bmatrix}s^2+15s/7+6/7 & 0 & 58/7\\ 0 & s^3+2s^2-3s+20 & 0\\ 3s/7-17/7 & 0 & s-1/7\end{bmatrix}, \tag{62}$$

$$Q(s) = \begin{bmatrix}s+22/7 & 5\\ 2s^2+s+1 & 0\\ 3/7 & 1\end{bmatrix}. \tag{63}$$

The free response of H is given by the solution of $P(D)\boldsymbol{y}_{\mathtt{hom}}(t) = 0$ (cf. (6)). A minimal state-space realization $\{A, B, C\}$ of $H(s)$ can be computed (cf. [9]) and then $J_{\mathtt{r}} = T_{\mathtt{r}}^{-1}AT_{\mathtt{r}}$ and $[KL] = CT_{\mathtt{r}}$ are determined by a suitable similarity transformation $T_{\mathtt{r}}$ (cf. Section IV):

$$J_{\mathtt{r}} = \begin{bmatrix}-4 & 0 & 0 & 0 & 0 & 0\\ 0 & -4 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 2 & 0 & 0\\ 0 & 0 & -2 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 2\\ 0 & 0 & 0 & 0 & -2 & 1\end{bmatrix}, \tag{64}$$

$$K = \begin{bmatrix}0 & -1\\ 1 & 0\\ 0 & 1\end{bmatrix},\ L = \begin{bmatrix}1 & 0 & 0 & 1\\ 1 & 0 & 0 & 0\\ 0 & -1 & 1 & 0\end{bmatrix}. \tag{65}$$

From Theorem 2 ($l = 1$, $\mu_1 = 1$, $k = 1$, $\nu_1 = 1$)

$$\boldsymbol{y}_{\mathtt{hom}}(t) = K_1\Gamma_{11}\boldsymbol{r}_1e^{-4t} + L_1\Psi_{11}\boldsymbol{c}_1e^t\cos(2t) + L_1\widetilde{\Psi}_{11}\boldsymbol{c}_1e^t\sin(2t)$$

and from (34), (35) and (39)-(42)

$$\boldsymbol{y}_{\mathtt{hom}}(t) = \begin{bmatrix}0 & -1\\ 1 & 0\\ 0 & 1\end{bmatrix}\boldsymbol{r}_1e^{-4t} + \begin{bmatrix}1 & 0 & 0 & 1\\ 1 & 0 & 0 & 0\\ 0 & -1 & 1 & 0\end{bmatrix}\boldsymbol{c}_1e^t\cos(2t) + \begin{bmatrix}0 & 1 & -1 & 0\\ 0 & 1 & 0 & 0\\ 1 & 0 & 0 & 1\end{bmatrix}\boldsymbol{c}_1e^t\sin(2t).$$

By setting $\boldsymbol{r}_1 := [\alpha_1\alpha_2]^T \in \mathbb{R}^2$, $\boldsymbol{c}_1 := [\alpha_3\alpha_4\alpha_5\alpha_6]^T \in \mathbb{R}^4$ an explicit parameterization of the free response is

$$\boldsymbol{y}_{\mathtt{hom}}(t) = \begin{bmatrix}-\alpha_2\\ \alpha_1\\ \alpha_2\end{bmatrix}e^{-4t} + \begin{bmatrix}\alpha_3+\alpha_6\\ \alpha_3\\ -\alpha_4+\alpha_6\end{bmatrix}e^t\cos(2t) + \begin{bmatrix}\alpha_4-\alpha_5\\ \alpha_4\\ \alpha_3+\alpha_6\end{bmatrix}e^t\sin(2t). \tag{66}$$

In $\boldsymbol{y}_{\mathtt{hom}}(t)$ the modes e^{-4t} and $\{e^t\cos(2t), e^t\sin(2t)\}$ are respectively associated to the output (mode) subspaces (cf. Definition 4):

$$\mathcal{F}_{11} = \operatorname{im} K_1\Gamma_{11} = \operatorname{im}\begin{bmatrix}0 & -1\\ 1 & 0\\ 0 & 1\end{bmatrix},\ \mathcal{G}_{11} = \operatorname{im} L_1\Psi_{11} = \mathbb{R}^3.$$

A finer structure in $\boldsymbol{y}_{\mathtt{hom}}(t)$ can be revealed by applying Proposition 2 (the system matrix A or $J_{\mathtt{r}}$ is diagonalizable):

$$\boldsymbol{y}_{\mathtt{hom}}(t) = \begin{bmatrix}-\alpha_2\\ \alpha_1\\ \alpha_2\end{bmatrix}e^{-4t} + \gamma_{12}e^t\left\{\begin{bmatrix}0\\0\\1\end{bmatrix}\cos(2t+\delta_{12}) + \begin{bmatrix}1\\0\\0\end{bmatrix}\cos(2t+\delta_{12}+\pi/2)\right\}$$

with $\alpha_1, \alpha_2 \in \mathbb{R}$, $\gamma_{12} \geq 0$ and $\delta_{12} \in [0, 2\pi)$.

VII. Conclusions

A new closed-form expression of the free response of multivariable linear systems has been presented. This expression is given by a linear combination of the system modes (associated to the roots of the *pole minimal polynomial*) and their vector coefficients belong to the *output mode subspaces*. The new finding is a system theory result that may be useful for e.g. modal analysis of structures [15] and time-domain identification techniques [16], [17].

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