On the equivalence of model inversion architectures for control applications

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***Abstract*— The main inversion-based control architectures are the plant and closed-loop inversion architectures. For scalar continuous-time linear systems, these architectures are shown to be fully equivalent for both the minimum and nonminimum-phase cases when exact stable inverses are used. This equivalence, deduced by using a behavioral approach, dictates that the two architectures deliver the same performances for any disturbance and mis-modeling affecting the controlled plant. A simulation example highlights that the equivalence still holds in practice when a careful truncation of the preaction control is performed.**

I. INTRODUCTION

Feedfoward control can improve the performances of (feedback) control systems. When precise output trajectory tracking is required, inversion-based control may be a very useful feedforward technique.

Given a desired output trajectory, an inversion-based controller computes the (model-based) inverse input which, in absence of modeling errors and disturbances, allows to precisely track the desired output trajectory. For linear minimum-phase systems the inversion-based controller simply provides the *standard inverse* input based on the straightforward use of the system's transfer function reciprocal. However, for nonminimum-phase systems the standard inverse is an unbounded signal because the (standard) inverse system is unstable. Nevertheless, also for nonminimum-phase systems there exists a bounded solution to the input-output inversion problem [1], [2]. Actually, this solution is a noncausal signal called the *stable inverse* input.

Inversion-based controllers cannot compensate for tracking errors caused by modeling inaccuracies or disturbances [3]. In practice, a feedback controller is always needed either for increasing the control system robustness or because the plant has to be stabilized. The structure in which the inversion-based and feedback controllers are arranged is referred to as an inversion architecture. There are two main inversion architectures: the plant inversion architecture [3], [4] (cf. Figure 1) and the closed-loop inversion architecture [5], [6] (cf. Figure 2). In absence of modeling errors and disturbances they both yield the same output, the desired output trajectory. However, in practice, modeling errors due to the plant's uncertainties and perturbations and disturbances always occur. Hence, in both architectures the actual plant's output differs from the desired one.

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As a consequence, from the viewpoint of the control applications, the relevant question is: which one of the two architectures performs better? Comparisons addressing this question have appeared in the control literature for scalar (single-input single-output) discrete-time linear systems [7], [8], [9].

In [7], for the minimum-phase case (the nominal plant and the feedback controller are both minimum-phase) an algebraic analysis shows both architectures to be equivalent in presence of uncertain plant dynamics. For the nonminimum-phase case (more specifically, the plant is nonminimum-phase whereas the controller is minimum-phase) the comparison shows that the two architectures perform differently when approximate stable inverses are used. For settle time applications (i.e. set-point regulation) the closed-loop inversion architecture appears to achieve superior performances. Further comparisons, still using approximate stable inverses, confirm the better performance of the closed-loop inversion architecture [8], [9]. However these comparisons seem to be not conclusive nor fully general because they depend on specific applications. As a consequence, the plant inversion architecture has still been used as reference architecture in several subsequent studies e.g. [10], [11].

For scalar continuous-time linear systems, this paper shows that the two architectures are fully equivalent when *exact* stable inverses are used as feedforward controllers. For the nonminimum-phase case — in which both the plant and the controller can be nonminimum-phase — an equivalence result (Theorem 1) is established for any desired output and any disturbance and mis-modeling affecting the plant (cf. Assumptions 1, 2, and 3). Crucial and instrumental in the deduction of the equivalence theorem is the adopted behavioral approach [12], [13]. Indeed, in this approach signals are naturally noncausal. Moreover, the key concept of *forced response of a system initially at rest at time* $-\infty$ has been added to the simplified behavior theory presented in [13] (cf. Sections II and III).

The paper is organized as follows. The behavioral approach to inversion-based control is summarized in Section II. Section III introduces the concept of forced response starting from time $-\infty$. In Section IV, the plant and closed-loop inversion architectures are presented (Subsection IV-A) and comparisons are made in Subsection IV-B and IV-D. A simulative example is included in Section V. For brevity, all the formal proofs of new propositions and results are omitted. The only exception is the provided proof of the equivalence theorem (Theorem 1 in Section IV), the paper's main result.

Notation: Given a real function $f : \mathbb{R} \to \mathbb{R}$, the following shorthand notations are used: $f(t^+) := \lim_{v \to t^+} f(v)$, $f(-\infty) := \lim_{t \to -\infty} f(t)$, $f(+\infty) := \lim_{t \to +\infty} f(t)$. We say that f is a *causal* signal when $f(t) = 0$, $t < 0$. The symbol $\equiv$ denotes an identity that holds over $\mathbb{R}$: $f \equiv 0$, i.e. $f(t) = 0$, $t \in \mathbb{R}$; $f \equiv g$, i.e. $f(t) = g(t)$, $t \in \mathbb{R}$. The i-th derivative of f is denoted by $f^{(i)}$ or $D^i f$ with D being the derivative operator. The Laplace transform of f is $F(s) := \mathcal{L}[f(t)]$. The analytical extension over $\mathbb{R}$ of the inverse Laplace transform is denoted by $\mathcal{L}^{-1}_{\mathtt{ae}}[\cdot]$ (defined by analytic continuation for negative times).

All the introduced polynomials have real coefficients. Given a polynomial $p(s)$, $\deg p$ is its degree, $p(D)$ denotes the corresponding differential operator, and it is said to be Hurwitz if all its roots have negative real parts. The *left half-plane* (LHP) and *right half-plane* (RHP) denote the sets of complex numbers having negative and positive real parts respectively. A linear, time-invariant system is said to be *minimum-phase* if all its zeros lie on the LHP or there are no (finite) zeros at all. It is said to be *nonminimum-phase* if there exists a system's zero on the RHP. System's zeros that lie on the RHP are said *nonminimum-phase* (or *unstable*) *zeros*.

II. PRELIMINARIES

A. System's behavioral presentation

Consider a linear time-invariant continuous-time system H defined by its transfer function

$$H(s) = \frac{b(s)}{a(s)} = \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_0}{a_n s^n + a_{n-1}s^{n-1} + \cdots + a_0}. \quad (1)$$

Polynomials $a(s)$ and $b(s)$ (with $\deg a = n \geq \deg b = m$) are coprime and $b(s)$ has no roots on the imaginary axis (the zero dynamics of H is hyperbolic). The *relative degree* of H is $r := n - m$. The scalar input and output of H, u and y respectively, belong to $C^\infty_{\mathtt{p}}(\mathbb{R})$, the set of piecewise C^∞-functions [13].

The *behavior* of H, denoted by $\mathcal{B}_H$ is the set of ordered pairs $(u, y) \in C^\infty_{\mathtt{p}}(\mathbb{R})$ that are *weak solutions* of the differential equation

$$\sum_{i=0}^{n} a_i D^i y = \sum_{i=0}^{m} b_i D^i u \ . \quad (2)$$

In the space $C^\infty_{\mathtt{p}}(\mathbb{R})^2$, a pair (u, y) is a weak solution of (2) it there exists a polynomial $g(t)$ with $\deg g \leq n-1$ such that the integral equation [13]

$$\sum_{i=0}^{n} a_i \int^{n-i} y(t) = \sum_{i=0}^{m} b_i \int^{n-i} u(t) + g(t) \quad (3)$$

is satisfied for all $t \in \mathbb{R}$. A relevant property of the behavior $\mathcal{B}_H$ is the following ($C^{-1} := C^\infty_{\mathtt{p}}(\mathbb{R})$).

***Proposition** 1 ([13]):* Consider a pair $(u, y) \in \mathcal{B}_H$ and let $p \in \mathbb{Z}$ with $p \geq -1$. Then $u \in C^p$ if and only if $y \in C^{r+p}$.

The pole and zero modes of H can be introduced as follows ([13]).

***Definition** 1 (pole and zero modes):* Given a real (complex) pole of H, $p \in \mathbb{R}$ ($p = \sigma \pm j\omega \in \mathbb{C}$) with multiplicity μ, the associated modes are: e^{pt}, $te^{p}t$, $\ldots$, $t^{\mu-1}e^{pt}$ ($e^{\sigma t}\cos(\omega t)$, $e^{\sigma t}\sin(\omega t)$, $\ldots$, $t^{\mu-1}e^{\sigma t}\cos(\omega t)$, $t^{\mu-1}e^{\sigma t}\sin(\omega t)$). All the pole modes of H are denoted by $m^{\mathtt{p}}_i(t)$, $i = 1, \ldots, n$.
Given a real (complex) zero of H, $z \in \mathbb{R}$ ($z = \rho \pm j\psi \in \mathbb{C}$) with multiplicity ν, the associated modes are: e^{zt}, $te^{z}t$, $\ldots$, $t^{\nu-1}e^{zt}$ ($e^{\rho t}\cos(\psi t)$, $e^{\rho t}\sin(\psi t)$, $\ldots$, $t^{\nu-1}e^{\sigma t}\cos(\omega t)$, $t^{\mu-1}e^{\sigma t}\sin(\omega t)$). All the zero modes of H are denoted by $m^{\mathtt{z}}_i(t)$, $i = 1, \ldots, m$.
Pole and zero modes can be classified into stable and unstable ones: $m^{\mathtt{p}-}_i(t)$, $m^{\mathtt{z}-}_j(t)$ and $m^{\mathtt{p}+}_k(t)$, $m^{\mathtt{z}+}_l(t)$ respectively ($m^{\mathtt{p}-}_i(+\infty) = 0$, $m^{\mathtt{z}-}_j(+\infty) = 0$ and $m^{\mathtt{p}+}_k(-\infty) = 0$, $m^{\mathtt{z}+}_l(-\infty) = 0$).

By polynomial division of $b(s)$ by $a(s)$, we rewrite the system's transfer function as

$$H(s) = h_{\mathtt{hfg}} + H_{\mathtt{dy}}(s)$$

where $h_{\mathtt{hfg}} := \lim_{s \to \infty} H(s)$ is the system's high-frequency gain [14] and $H_{\mathtt{dy}}(s)$ is a strictly proper rational function representing the (purely) dynamic part of H. (Note that if the relative degree $r \geq 1$ then $h_{\mathtt{hfg}} = 0$ and $H(s) = H_{\mathtt{dy}}(s)$.) Then define $h_{\mathtt{dy}}(t) := \mathcal{L}^{-1}_{\mathtt{ae}}[H_{\mathtt{dy}}(s)]$ and a useful characterization of the behavior set $\mathcal{B}_H$ is the following (cf. [13])

***Proposition** 2 (Input-output representation of $\mathcal{B}_H$):*

$$\mathcal{B}_H = \{(u, y) \in C^\infty_{\mathtt{p}}(\mathbb{R})^2 : y(t) = h_{\mathtt{hfg}} u(t) + \int_0^t h_{\mathtt{dy}}(t - v)u(v)dv + \sum_{i=1}^{n} f_i m^{\mathtt{p}}_i(t), t \in \mathbb{R}; f_i \in \mathbb{R}\}. \quad (4)$$

By polynomial division, write $a(s) = q(s)b(s) + c(s)$ with $\deg c \leq m - 1$ so that the transfer function inverse becomes

$$H^{-1}(s) = q(s) + H_0(s) \quad (5)$$

with $H_0(s) := c(s)/b(s)$ representing the zero dynamics of H. Define $h_0(t) := \mathcal{L}^{-1}_{\mathtt{ae}}[H_0(s)]$ and introduce the output-input representation of the behavior $\mathcal{B}_H$ [13].

***Proposition** 3 (Output-input representation of $\mathcal{B}_H$):*

$$\mathcal{B}_H = \{(u, y) \in C^\infty_{\mathtt{p}}(\mathbb{R})^2 : y \in C^{r-1},\ u(t) = q(D)y(t^+) + \int_0^t h_0(t - v)y(v)dv + \sum_{i=1}^{m} g_i m^{\mathtt{z}}_i(t), t \in \mathbb{R}; g_i \in \mathbb{R}\}. \quad (6)$$

B. Stable input-output inversion

Let be given a causal desired output $y_{\mathtt{d}}$ of H and assume that $y_{\mathtt{d}} \in C^\infty_{\mathtt{p}}(\mathbb{R}) \cap C^{r-1}$ and $y_{\mathtt{d}}$, $y^{(1)}_{\mathtt{d}}$, $\ldots$, $y^{(r)}_{\mathtt{d}}$ are all bounded time-functions on $\mathbb{R}$. The *standard inverse input* can be expressed as (cf. Proposition 3)

$$u'_{H,\mathtt{d}}(t) = q(D)y_{\mathtt{d}}(t^+) + \int_0^t h_0(t - v)y_{\mathtt{d}}(v)dv,\ t \in \mathbb{R}. \quad (7)$$

As known, if H is nonminimum-phase $u'_{H,\mathtt{d}}(t)$ diverges exponentially because the zero dynamics $H_0(s)$ is unstable [1]. In this case $u'_{H,\mathtt{d}}(t)$ cannot be used for control applications. Nevertheless, a bounded noncausal inverse input exists [1], [2]. By partial fraction decomposition, $H_0(s)$ can be split as

$$H_0(s) = H^-_0(s) + H^+_0(s) \quad (8)$$

where $H_0^-(s)$ and $H_0^+(s)$ represent the stable and unstable zero dynamics respectively. Hence, define $h_0^-(t) := \mathcal{L}_{\mathtt{ae}}^{-1}[H_0^-(s)]$, $h_0^+(t) := \mathcal{L}_{\mathtt{ae}}^{-1}[H_0^+(s)]$ so that

$$h_0(t) = h_0^-(t) + h_0^+(t),\ t \in \mathbb{R} \tag{9}$$

and the bounded noncausal inverse input — called the *stable inverse input* — can be expressed as [13]

$$u_{H,\mathtt{d}}(t) = q(D)y_{\mathtt{d}}(t^+) + \int_0^t h_0^-(t-v)y_{\mathtt{d}}(v)dv - \int_t^{+\infty} h_0^+(t-v)y_{\mathtt{d}}(v)dv,\ t \in \mathbb{R}. \tag{10}$$

A relationship between $u'_{H,\mathtt{d}}$ and $u_{H,\mathtt{d}}$ can be established (cf. [15]).

***Proposition** 4:* The stable inverse $u_{H,\mathtt{d}}$ can be expressed as

$$u_{H,\mathtt{d}}(t) = u'_{H,\mathtt{d}}(t) + u_{H,\mathtt{zm}}(t) \quad t \in \mathbb{R}, \tag{11}$$

where $u_{H,\mathtt{zm}}(t) = -\int_0^{+\infty} h_0^+(t-v)y_d(v)dv$ is a (linear) combination of the system's unstable zero modes.

***Remark** 1:* Note that the standard inverse $u'_{H,\mathtt{d}}$ is always a causal signal whereas the stable inverse $u_{H,\mathtt{d}}$ is noncausal if H is nonminimum-phase. Hence, relation (11) implies that $u_{H,\mathtt{d}}$ exhibits the *preaction* (or *preactuation*) *control* phenomenon: the inverse $u_{H,\mathtt{d}}$ is not identically zero on negative times — actually it is a linear combination of unstable zero modes — whereas the corresponding desired output $y_{\mathtt{d}}$ is identically zero (on negative times) [1], [13].

III. THE SYSTEM'S FORCED RESPONSE FROM TIME $-\infty$

The following definitions are introduced.

***Definition** 2 (System initially at rest):* Let be given a system H with input u and output y, i.e. $(u, y) \in \mathcal{B}_H$ (cf. Subsection II-A). H is said to be initially at rest (at time $-\infty$) if $u(-\infty) = 0$ and $y(-\infty) = 0$.

***Definition** 3 (Forced response from time $-\infty$):* Assume that H is asymptotically stable. Let be given a pair $(u, y) \in \mathcal{B}_H$ with $u(-\infty) = 0$ and $y(-\infty) = 0$. Then y is called a forced response from time $-\infty$ or more precisely a forced response to (input) u of H initially at rest (at time $-\infty$).

A key result is the following.

***Proposition** 5 (Uniqueness of the forced response):* Assume that H is asymptotically stable and let be given a pair $(u, y) \in \mathcal{B}_H$ with $u(-\infty) = 0$ and $y(-\infty) = 0$. Then, y is the unique forced response to input u of H initially at rest (at time $-\infty$).

Given an input $u \in C_{\mathtt{p}}^{\infty}(\mathbb{R})$ with $u(-\infty) = 0$ the computation of the forced response from time $-\infty$ is addressed by the following result.

***Proposition** 6 (Forced response operator):* Assume that H is asymptotically stable. Let be given an input $u \in C_{\mathtt{p}}^{\infty}(\mathbb{R})$ with $u(-\infty) = 0$ and introduce the following operator:

$$H(u)(t) := h_{\mathtt{hfg}}u(t) + \int_{-\infty}^t h_{\mathtt{dy}}(t-v)u(v)dv,\ t \in \mathbb{R}. \tag{12}$$

Then $(u, H(u)) \in \mathcal{B}$ and $H(u)(-\infty) = 0$, i.e. $H(u)(t)$ is the forced response to input u of H initially at rest at time $-\infty$.

***Remark** 2:* With a slight abuse of notation we denote the forced response operator introduced in (12) with the same symbol H used to denote the system itself and its transfer function $H(s)$. The context clarifies what H stands for.

A related technical result that does not require the system's asymptotic stability follows.

***Lemma** 1:* Assume that H is initially at rest at time $-\infty$ with $(u, y) \in \mathcal{B}_H$. The input u is identically zero, i.e. $u \equiv 0$, and the output y is bounded over $\mathbb{R}$. Then $y \equiv 0$, i.e. the output is identically zero too.

IV. COMPARISON OF MODEL INVERSION CONTROL ARCHITECTURES

A. The inversion-based control architectures

In the framework of inversion-based control, consider a plant to be controlled whose nominal transfer function is $P(s)$ and its relative degree is r_P. The actual plant that may differ from the nominal plant due to model inaccuracies and perturbations is denoted by $\tilde{P}$ and its transfer function is $\tilde{P}(s)$. The set of all possible perturbed plants $\tilde{P}$ belongs to $\mathcal{P}$, the *uncertain plant set* (also $P \in \mathcal{P}$).

There are two main model inversion control architectures (also called schemes in the following): (1) the *plant inversion* architecture [3], [4] and (2) the *closed-loop inversion* architecture [5], [6]. They are depicted by block diagrams in Figure 1 and 2 respectively. Both architectures use a

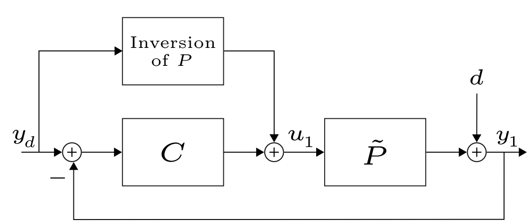

Fig. 1. Plant inversion architecture or scheme 1.

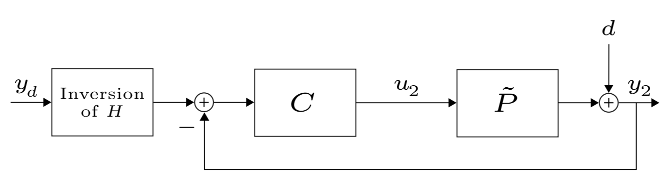

Fig. 2. Closed-loop inversion architecture or scheme 2 in which $H := (1+CP)^{-1}CP$.

feedback controller C that is designed to ensure the internal asymptotic stability of the closed-loop system for any $\tilde{P} \in \mathcal{P}$. Thus, the following assumption is considered [16].

***Assumption** 1:* For any $\tilde{P} \in \mathcal{P}$: (i) all the roots of $1 + C(s)\tilde{P}(s) = 0$ have negative real parts; (ii) there are no pole-zero cancellations between $C(s)$ and $\tilde{P}(s)$ on the closed RHP.

Further assumptions are the following.

Assumption *2:* The zero dynamics of the nominal plant P and the controller C are both hyperbolic, i.e. there are no zero of $P(s)$ or $C(s)$ on the imaginary axis of the complex plane.

Assumption *3:* The disturbance $d \in C^{\infty}_{\mathtt{p}}(\mathbb{R})$ is bounded over $\mathbb{R}$.

The nominal closed-loop transfer function is $H(s) := [1 + C(s)P(s)]^{-1}C(s)P(s)$ and its relative degree is $r = r_C + r_P$ with r_C being the relative degree of C.

The causal desired output is $y_{\mathtt{d}} \in C^{\infty}_{\mathtt{p}}(\mathbb{R}) \cap C^{r-1}$ and assume that $y_{\mathtt{d}}$ and its derivatives $y^{(1)}_{\mathtt{d}}, \ldots, y^{(r)}_{\mathtt{d}}$ are all bounded time-functions (cf. Subsection II-B). The inverse inputs applied to H that cause $y_{\mathtt{d}}$ are the standard inverse $u'_{H,\mathtt{d}}$ and the stable inverse $u_{H,\mathtt{d}}$. Their closed-form expressions are reported in (7) and (10).

The nominal plant and the controller transfer function are written as

$$P(s) = \frac{b_P(s)}{a_P(s)} \text{ and } C(s) = \frac{b_C(s)}{a_C(s)} \tag{13}$$

with $a_P(s)$, $b_P(s)$ and $a_C(s)$, $b_C(s)$ being both coprime polynomial pairs. By Assumption 2, polynomials $b_P(s)$ and $b_C(s)$ can be factorized as

$$b_P(s) = b^-_P(s)b^+_P(s) \text{ and } b_C(s) = b^-_C(s)b^+_C(s) \tag{14}$$

with $b^-_P(s)$, $b^-_C(s)$, $b^+_P(-s)$, and $b^+_C(-s)$ being all Hurwitz polynomials.

The input-output inversion of the nominal plant follows (cf. Subsection II-B). By polynomial division $a_P(s) = q_P(s)b_P(s)+c_P(s)$ with $\deg c_P < \deg b_P$ so that the plant's transfer function inverse is expressed by

$$P^{-1}(s) = q_P(s) + P_0(s) \tag{15}$$

with $P_0(s) = \frac{c_P(s)}{b_P(s)}$ representing the plant's zero dynamics. $P_0(s)$ can be split into stable and unstable parts by partial fraction expansion as

$$P_0(s) = \frac{c_P(s)}{b^-_P(s)b^+_P(s)} = P^-_0(s) + P^+_0(s) \tag{16}$$

having set $P^-_0(s) = \frac{c^-_P(s)}{b^-_P(s)}$, $P^+_0(s) = \frac{c^+_P(s)}{b^+_P(s)}$ with c^-_P and c^+_P being suitable polynomials. Define $p_0(t) := \mathcal{L}^{-1}_{\mathtt{ae}}[P_0(s)]$, $p^-_0(t) := \mathcal{L}^{-1}_{\mathtt{ae}}[P^-_0(s)]$, and $p^+_0(t) := \mathcal{L}^{-1}_{\mathtt{ae}}[P^+_0(s)]$ so that the standard inverse input and the stable inverse one can be expressed respectively as ($t \in \mathbb{R}$)

$$u'_{P,\mathtt{d}}(t) = q_P(D)y_{\mathtt{d}}(t^+) + \int_0^t p_0(t-v)y_{\mathtt{d}}(v)dv, \tag{17}$$

$$u_{P,\mathtt{d}}(t) = q_P(D)y_{\mathtt{d}}(t^+) + \int_0^t p^-_0(t-v)y_{\mathtt{d}}(v)dv - \int_t^{+\infty} p^+_0(t-v)y_{\mathtt{d}}(v)dv. \tag{18}$$

Remark *3:* Note that $y_{\mathtt{d}}$ has a continuity order sufficiently high to be inverted on the nominal plant P (cf. [13]). Indeed, $y_{\mathtt{d}} \in C^{r-1}$ implies $y_{\mathtt{d}} \in C^{r_P-1}$ because $r \geq r_P$ ($r_C \geq 0$).

B. The minimum-phase case

In case the control architectures 1 and 2 (cf. Figures 1 and 2) are used with standard inverses the following result follows.

Lemma *2:* Assume the control architectures 1 and 2 are at rest at time $-\infty$ and apply the desired output $y_{\mathtt{d}}$ and the standard inverses $u'_{P,\mathtt{d}}$ and $u'_{H,\mathtt{d}}$ respectively. Then $u_1 \equiv u_2$ and $y_1 \equiv y_2$ for any perturbed plant $\tilde{P} \in \mathcal{P}$ and any disturbance d.

When both the nominal plant and the controller are minimum-phase the stable inverse inputs $u_{P,\mathtt{d}}$ and $u_{H,\mathtt{d}}$ coincide with the standard inverses $u'_{P,\mathtt{d}}$ and $u'_{H,\mathtt{d}}$ respectively, i.e. $u_{P,\mathtt{d}} \equiv u'_{P,\mathtt{d}}$ and $u_{H,\mathtt{d}} \equiv u'_{H,\mathtt{d}}$. Hence, Lemma 2 immediately implies the following equivalence result.

Proposition *7:* Let P and C be minimum-phase systems. Assume the control architectures 1 and 2 are at rest at time $-\infty$ and apply the desired output $y_{\mathtt{d}}$ and the stable inverses $u_{P,\mathtt{d}}$ and $u_{H,\mathtt{d}}$ respectively. Then $u_1 \equiv u_2$ and $y_1 \equiv y_2$ for any perturbed plant $\tilde{P} \in \mathcal{P}$ and any disturbance d.

Proposition 7 means that when P and C are minimum-phase, the plant and the closed-loop inversion architectures deliver the same performance regardless of uncertainties, perturbations, and disturbances on the plant. Essentially, this result was originally stated by [7] for discrete-time systems.

C. Algebraic results

Factorize the polynomial $b^+_C(s)$ (cf. (14)) as

$$b^+_C(s) = b^+_{CC}(s)b^+_{CP}(s) \tag{19}$$

such that b^+_{CC} and b^+_{CP} are coprime and b^+_{CP} has as roots all the common roots of b^+_C and b^+_P (i.e. if z^+ satisfies $b^+_C(z^+) = 0$, $b^+_P(z^+) = 0$ then $b^+_{CP}(z^+) = 0$). Hence, the closed-loop's unstable zero dynamics $H^+_0(s)$ can be split up as (cf. (5), (8) and (13), (14))

$$H^+_0(s) = \frac{c^+(s)}{b^+_{CC}(s)b^+_{CP}(s)b^+_P(s)} = H^+_{0,CC}(s) + H^+_{0,P}(s) \tag{20}$$

with

$$H^+_{0,CC}(s) := \frac{d^+_{CC}(s)}{b^+_{CC}(s)}, \;\; H^+_{0,P}(s) := \frac{d^+_P(s)}{b^+_{CP}(s)b^+_P(s)} \tag{21}$$

and c^+, d^+_{CC}, and d^+_P being suitable polynomials. $H^+_{0,CC}$ represents the closed-loop's unstable zero dynamics due to the nonminimum-phase zeros of C that do not coincide with any zero of P. On the other hand, $H^+_{0,P}$ represents the closed-loop's unstable zero dynamics due to the nonminimum-phase zeros of P that may have increased multiplicity due to the possible presence of common nonminimum-phase zeros of C and P, i.e. the roots of b^+_{CP}. Define $h^+_{0,CC}(t) := \mathcal{L}^{-1}_{\mathtt{ae}}[H^+_{0,CC}(s)]$, $h^+_{0,P}(t) := \mathcal{L}^{-1}_{\mathtt{ae}}[H^+_{0,P}(s)]$ and the following identity follows:

$$h^+_0(t) = h^+_{0,CC}(t) + h^+_{0,P}(t), \; t \in \mathbb{R}. \tag{22}$$

The fundamental algebraic result leading to Theorem 1 is the following.

Lemma 3: There exists a polynomial $\lambda(s)$ such that

$$a_C(s)P_0^+(s) - b_C(s)H_{0,P}^+(s) = \lambda(s)\ . \qquad (23)$$

In the rest of the paper we assume for simplicity that all nonminimum-phase zeros of C and P are simple and real. With this assumption, let $z_{PP,i}^+$, $i = 1,\dots,m_{PP}^+$ denote the nonminimum-phase zeros of P which do not coincide with any of the zeros of C. Furthermore, let $z_{CP,i}^+$, $i = 1,\dots,m_{CP}^+$ and $z_{CC,i}^+$, $i = 1,\dots,m_{CC}^+$ denote the roots of $b_{CP}^+(s)$ and $b_{CC}^+(s)$ respectively. The above identity (23) permits to derive the next algebraic result.

Lemma 4: Let $P_0^+(s)$ and $H_{0,P}^+(s)$ be expressed as

$$P_0^+(s) = \sum_{i=1}^{m_{PP}^+} \frac{\alpha_{PP,i}}{s - z_{PP,i}^+} + \sum_{i=1}^{m_{CP}^+} \frac{\alpha_{CP,i}}{s - z_{CP,i}^+}, \qquad (24)$$

$$H_{0,P}^+(s) = \sum_{i=1}^{m_{PP}^+} \frac{\gamma_{PP,i}}{s - z_{PP,i}^+} + \sum_{i=1}^{m_{CP}^+} \left(\frac{\gamma_{CP,i,0}}{s - z_{CP,i}^+} + \frac{\gamma_{CP,i,1}}{(s - z_{CP,i}^+)^2} \right). \qquad (25)$$

Then

$$\alpha_{PP,i} a_C(z_{PP,i}^+) = \gamma_{PP,i} b_C(z_{PP,i}^+), \quad i = 1,\dots,m_{PP}^+,$$
$$\alpha_{CP,i} a_C(z_{CP,i}^+) = \gamma_{CP,i,1} b_{C,i}(z_{CP,i}^+), \quad i = 1,\dots,m_{CP}^+, \qquad (26)$$

where $b_{C,i}(s)$, $i = 1,\dots,m_{CP}^+$ are polynomials defined by $b_{C,i}(s) := b_C(s)/(s - z_{CP,i}^+)$.

D. The nonminimum-phase case: main result

The equivalence of the inversion-based control architectures (schemes 1 and 2) asserted by Proposition 7 in the restricted minimum-phase case is extended to nonminimum-phase plants and controllers. This is the main finding stated by the following equivalence theorem.

Theorem 1: Assume the control architectures 1 and 2 are at rest at time $-\infty$ and apply the desired output $y_{\mathtt{d}}$ and the stable inverses $u_{P,\mathtt{d}}$ and $u_{H,\mathtt{d}}$ respectively. Then $u_1 \equiv u_2$ and $y_1 \equiv y_2$ for any perturbed plant $\tilde{P} \in \mathcal{P}$ and any disturbance d.

Proof: By virtue of Proposition 4, the stable inverse inputs $u_{P,\mathtt{d}}$ and $u_{H,\mathtt{d}}$ (applied to schemes 1 and 2 respectively) can be written as

$$u_{P,\mathtt{d}}(t) = u'_{P,\mathtt{d}}(t) + u_{P,\mathtt{zm}}(t)\ , t \in \mathbb{R} \qquad (27)$$

$$u_{H,\mathtt{d}}(t) = u'_{H,\mathtt{d}}(t) + u_{H,\mathtt{zm}}(t)\ , t \in \mathbb{R} \qquad (28)$$

where $u_{P,\mathtt{zm}}(t) := -\int_0^{+\infty} p_0^+(t-v) y_{\mathtt{d}}(v) dv$, $t \in \mathbb{R}$ and $u_{H,\mathtt{zm}}(t) := -\int_0^{+\infty} h_0^+(t-v) y_{\mathtt{d}}(v) dv$, $t \in \mathbb{R}$.

Furthermore, p_0^+ and h_0^+ can be expressed as (cf. (22), (24) and (25)):

$$p_0^+(t) = \sum_{i=1}^{m_{PP}^+} \alpha_{PP,i} e^{z_{PP,i}^+ t} + \sum_{i=1}^{m_{CP}^+} \alpha_{CP,i} e^{z_{CP,i}^+ t}, \quad t \in \mathbb{R} \qquad (29)$$

$$h_0^+(t) = \sum_{i=1}^{m_{CC}^+} \gamma_{CC,i} e^{z_{CC,i}^+ t} + \sum_{i=1}^{m_{PP}^+} \gamma_{PP,i} e^{z_{PP,i}^+ t} + \sum_{i=1}^{m_{CP}^+} \left(\gamma_{CP,i,0} e^{z_{CP,i}^+ t} + \gamma_{CP,i,1} t e^{z_{CP,i}^+ t} \right) \quad , t \in \mathbb{R}. \qquad (30)$$

As a consequence, $u_{P,\mathtt{zm}}$ and $u_{H,\mathtt{zm}}$ are linear combinations of unstable zero modes, $t \in \mathbb{R}$:

$$u_{P,\mathtt{zm}}(t) = -\sum_{i=1}^{m_{PP}^+} \alpha_{PP,i} f_{PP,i} e^{z_{PP,i}^+ t} - \sum_{i=1}^{m_{CP}^+} \alpha_{CP,i} f_{CP,i} e^{z_{CP,i}^+ t}, \qquad (31)$$

$$u_{H,\mathtt{zm}}(t) = -\sum_{i=1}^{m_{CC}^+} \gamma_{CC,i} g_i e^{z_{CC,i}^+ t} - \sum_{i=1}^{m_{PP}^+} \gamma_{PP,i} f_{PP,i} e^{z_{PP,i}^+ t} - \sum_{i=1}^{m_{CP}^+} \left[\gamma_{CP,i,0} f_{CP,i} + \gamma_{CP,i,1} \Big(f_{CP,i} t - f_{CP,i,1} \Big) \right] e^{z_{CP,i}^+ t}, \qquad (32)$$

with $f_{PP,i} := \int_0^{+\infty} e^{-z_{PP,i}^+ v} y_{\mathtt{d}}(v) dv$, $f_{CP,i} := \int_0^{+\infty} e^{-z_{CP,i}^+ v} y_{\mathtt{d}}(v) dv$, $g_i := \int_0^{+\infty} e^{-z_{CC,i}^+ v} y_{\mathtt{d}}(v) dv$ and $f_{CP,i,1} := \int_0^{+\infty} v e^{-z_{CP,i}^+ v} y_{\mathtt{d}}(v) dv$ which are well defined because $y_{\mathtt{d}}$ is bounded over $\mathbb{R}$.

The linearity of the schemes 1 and 2 makes it possible to determine the control inputs as follows:

$$u_1(t) = u_1(t)\Big|_{\substack{y_{\mathtt{d}}, u'_{P,\mathtt{d}}, d \\ u_{P,\mathtt{zm}} \equiv 0}} + u_1(t)\Big|_{\substack{y_{\mathtt{d}} \equiv 0, u'_{P,\mathtt{d}} \equiv 0, d \equiv 0 \\ u_{P,\mathtt{zm}}}}\ , \qquad (33)$$

$$u_2(t) = u_2(t)\Big|_{\substack{y_{\mathtt{d}}, u'_{H,\mathtt{d}}, d \\ u_{H,\mathtt{zm}} \equiv 0}} + u_2(t)\Big|_{\substack{y_{\mathtt{d}} \equiv 0, u'_{H,\mathtt{d}} \equiv 0, d \equiv 0 \\ u_{H,\mathtt{zm}}}}\ . \qquad (34)$$

Lemma 2 showed that $u_1\Big|_{\substack{y_{\mathtt{d}}, u'_{P,\mathtt{d}}, d \\ u_{P,\mathtt{zm}} \equiv 0}} \equiv u_2\Big|_{\substack{y_{\mathtt{d}}, u'_{H,\mathtt{d}}, d \\ u_{H,\mathtt{zm}} \equiv 0}}$, hence to prove that $u_1 \equiv u_2$ we need to ascertain

$$u_1(t)\Big|_{\substack{y_{\mathtt{d}} \equiv 0, u'_{P,\mathtt{d}} \equiv 0, d \equiv 0 \\ u_{P,\mathtt{zm}}}} = u_2(t)\Big|_{\substack{y_{\mathtt{d}} \equiv 0, u'_{H,\mathtt{d}} \equiv 0, d \equiv 0 \\ u_{H,\mathtt{zm}}}}\ , t \in \mathbb{R}. \qquad (35)$$

Under the assumption of $y_{\mathtt{d}} \equiv 0$, $u'_{P,\mathtt{d}} \equiv 0$, $u'_{H,\mathtt{d}} \equiv 0$, and $d \equiv 0$ the only external inputs in schemes 1 and 2 are $u_{P,\mathtt{zm}}$ and $u_{H,\mathtt{zm}}$ respectively. In this scenario, the transfer functions from $u_{P,\mathtt{zm}}$ to u_1 and from $u_{H,\mathtt{zm}}$ to u_2 are

$$T_1(s) := \frac{1}{1 + C(s)\tilde{P}(s)}\ ,\ T_2(s) := \frac{C(s)}{1 + C(s)\tilde{P}(s)} \qquad (36)$$

respectively. (Herein, to simplify the notation, u_1 and u_2 stand for $u_1\Big|_{\substack{y_{\mathtt{d}} \equiv 0, u'_{P,\mathtt{d}} \equiv 0, d \equiv 0 \\ u_{P,\mathtt{zm}}}}$ and $u_2\Big|_{\substack{y_{\mathtt{d}} \equiv 0, u'_{H,\mathtt{d}} \equiv 0, d \equiv 0 \\ u_{H,\mathtt{zm}}}}$ respectively.) Let $\tilde{P}(s) := \frac{b_{\tilde{P}}(s)}{a_{\tilde{P}}(s)}$ with $a_{\tilde{P}}(s)$, $b_{\tilde{P}}(s)$ suitable polynomials and the transfer functions in (36) can be expressed as

$$T_1(s) = \frac{a_{\tilde{P}}(s) a_C(s)}{a_{\mathtt{cl}}(s)}\ ,\ T_2(s) = \frac{a_{\tilde{P}}(s) b_C(s)}{a_{\mathtt{cl}}(s)} \qquad (37)$$

with $a_{\mathtt{cl}}(s) := a_C(s) a_{\tilde{P}}(s) + b_C(s) b_{\tilde{P}}(s)$ being an Hurwitz polynomial by Assumption 1. The control architectures are initially at rest, i.e. both C and $\tilde{P}$ are initially at rest, and $u_{P,\mathtt{zm}}(-\infty) = 0$, $u_{H,\mathtt{zm}}(-\infty) = 0$ (cf. (31), (32)).

Therefore $u_1(-\infty) = 0$, $u_2(-\infty) = 0$ and u_1, u_2 are the forced responses to inputs $u_{P,\mathtt{zm}}$, $u_{H,\mathtt{zm}}$ of T_1, T_2 at rest at time $-\infty$ respectively, i.e. $u_1(t) = T_1(u_{P,\mathtt{zm}})(t)$, $u_2(t) = T_2(u_{H,\mathtt{zm}})(t)$, $t \in \mathbb{R}$ (cf. Proposition 6).

The pairs $(u_{P,\mathtt{zm}}, u_1)$ and $(u_{H,\mathtt{zm}}, u_2)$ are weak solutions of the differential equations associated to T_1 and T_2 respectively, but since $u_{P,\mathtt{zm}}$, $u_{H,\mathtt{zm}} \in C^\infty$ and u_1, $u_2 \in C^\infty$ by virtue of Proposition 1 they are also *strong solutions* (cf. [13]), i.e.

$$a_{\mathtt{cl}}(D)u_1(t) = a_{\tilde{P}}(D)a_C(D)u_{P,\mathtt{zm}}(t),\ t \in \mathbb{R}, \tag{38}$$

$$a_{\mathtt{cl}}(D)u_2(t) = a_{\tilde{P}}(D)b_C(D)u_{H,\mathtt{zm}}(t),\ t \in \mathbb{R}. \tag{39}$$

Since $a_C(D)u_{P,\mathtt{zm}}(-\infty) = 0$, $b_C(D)u_{H,\mathtt{zm}}(-\infty) = 0$ from (38) and (39) it follows that u_1 and u_2 are the forced responses to input $a_C(D)u_{P,\mathtt{zm}}$ and $b_C(D)u_{H,\mathtt{zm}}$ respectively of the system initially at rest associated to the transfer function $\frac{a_{\tilde{P}}(s)}{a_{\mathtt{cl}}(s)}$.

Remark that given a polynomial $w(s)$ then $w(D)e^{zt} = w(z)e^{zt}$, $t \in \mathbb{R}$. Furthermore, if $s = z$ is a root of $w(s)$ then $w(D)e^{zt} \equiv 0$. Hence, the input $a_C(D)u_{P,\mathtt{zm}}$ can be expressed as (cf. (31)):

$$\begin{aligned} a_C(D)u_{P,\mathtt{zm}}(t) = &-\sum_{i=1}^{m_{PP}^+} \alpha_{PP,i} f_{PP,i} a_C(z_{PP,i}^+) e^{z_{PP,i}^+ t} \\ &-\sum_{i=1}^{m_{CP}^+} \alpha_{CP,i} f_{CP,i} a_C(z_{CP,i}^+) e^{z_{CP,i}^+ t}, \quad t \in \mathbb{R}. \end{aligned} \tag{40}$$

By taking into account that $b_C(D)te^{z_{CP,i}^+ t} = b_{C,i}(D)e^{z_{CP,i}^+ t}$, $i = 1, \dots, m_{CP}^+$ it follows (cf. (32))

$$\begin{aligned} b_C(D)u_{H,\mathtt{zm}}(t) = &-\sum_{i=1}^{m_{PP}^+} \gamma_{PP,i} f_{PP,i} b_C(z_{PP,i}^+) e^{z_{PP,i}^+ t} \\ &-\sum_{i=1}^{m_{CP}^+} \gamma_{CP,i,1} f_{CP,i} b_{C,i}(z_{CP,i}^+) e^{z_{CP,i}^+ t}, \quad t \in \mathbb{R}. \end{aligned} \tag{41}$$

The algebraic identities (26) of Lemma 4 prove that expressions (40) and (41) coincide so that $a_C(D)u_{P,\mathtt{zm}} \equiv b_C(D)u_{H,\mathtt{zm}}$. Hence, by Proposition 5 the forced responses u_1 and u_2 must be the same unique signal, i.e. $u_1 \equiv u_2$. It is then proved that inputs u_1 and u_2 are equal to each other under the application of $y_{\mathtt{d}}$, $u_{P,\mathtt{d}}$, and $u_{H,\mathtt{d}}$ for any perturbed plant $\tilde{P}$ and disturbance d.

The external inputs of schemes 1 and 2, i.e. $y_{\mathtt{d}}$, $u_{P,\mathtt{d}}$, $u_{H,\mathtt{d}}$, and d are all bounded over $\mathbb{R}$ (cf. Assumption 3). Hence, by the internal asymptotic stability property shared by both schemes (cf. Assumption 1), the outputs y_1 and y_2 are both bounded over $\mathbb{R}$. Moreover, since schemes 1 and 2 are initially at rest, all the signals are zero at time $-\infty$. Hence, the formal outputs of $\tilde{P}$ in these schemes, i.e. $y_1 - d$ and $y_2 - d$ respectively (the plant's outputs unaffected by the disturbance d), are both zero at time $-\infty$ as well as bounded over $\mathbb{R}$. Then $(u_1, y_1 - d)$, $(u_2, y_2 - d) \in \mathcal{B}_{\tilde{P}}$ so that from $u_1 \equiv u_2$ and by making the difference between these pairs we obtain $(0, y_2 - y_1) \in \mathcal{B}_{\tilde{P}}$ with $y_2 - y_1$ being bounded over $\mathbb{R}$. Hence, by Lemma 1 $y_2 - y_1 \equiv 0$, i.e. $y_1 \equiv y_2$. This concludes the proof. ■

V. A SIMULATION COMPARISON

Consider a nonminimum-phase plant affected by perturbations for which a set-point regulation problem is addressed by inversion-based control. Let the nominal plant be [17]

$$P(s) = -4\frac{(s-1)(s+1)}{(s+2)(s^2+s+2)}$$

and the desired output be a monotonically increasing function defined by a *transition polynomial* [18] [19] $y_{\mathtt{d}}(t) := 0$, $t < 0$, $y_{\mathtt{d}}(t) := -\frac{2}{27}t^3 + \frac{1}{3}t^2$, $t \in [0, 3]$, and $y_{\mathtt{d}}(t) := 1$, $t > 3$ ($y_{\mathtt{d}} \in C_{\mathtt{p}}^\infty(\mathbb{R}) \cap C^1$). For the simulative implementation we compare the plant inversion architecture (cf. Figure 1) with the closed-loop inversion architecture (cf. Figure 2). These schemes use the same feedback controller

$$C(s) = -7\frac{(s+2)(s^2+s+2)(s-2)}{s(s+1)(s+10)(s+20)}$$

that ensures the closed-loop internal stability for any perturbed plant belonging to a supposed uncertain set $\mathcal{P}$ (Assumption 1). Both the plant P and controller C have an hyperbolic zero dynamics (Assumption 2) and for simplicity, the disturbance d is set to be identically zero, i.e. $d \equiv 0$ (cf. Assumption 3). For the simulations, the following perturbed plant is considered

$$\tilde{P}(s) = -3.8\frac{(s-1.05)(s+0.97)}{(s+2.1)(s^2+1.05s+1.9)(0.01s+1)}$$

in which both parametric and high-frequency perturbations are present.

The stable inverse inputs $u_{P,\mathtt{d}}$ and $u_{H,\mathtt{d}}$ to be applied on the plant inversion scheme 1 and the closed-loop inversion scheme 2 respectively are determined by formulas (18) and (10). They are plotted in Figure 3. To perform the simulations

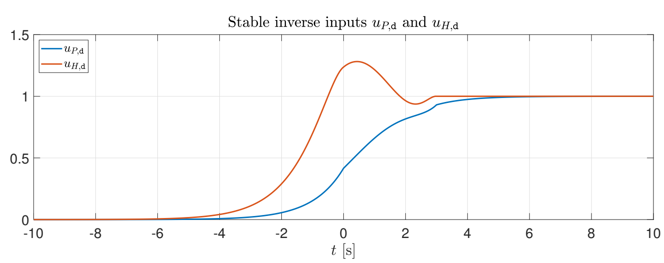

Fig. 3. Plots of the stable inverses $u_{P,\mathtt{d}}$ (in blue) and $u_{H,\mathtt{d}}$ (in red) for the plant and closed-loop inversion architectures respectively.

it is necessary to truncate the exact infinite preaction of the stable inverses $u_{P,\mathtt{d}}$ and $u_{H,\mathtt{d}}$ (cf. Remark 1). Indeed, a stable (noncausal) inverse input exponentially decays to zero as $t \to -\infty$ so that a *preaction time* $t_{\mathtt{pre}}$ can be determined for which the inverse input is almost identically zero on $(-\infty, -t_{\mathtt{pre}})$ and it significantly differs from zero on $[-t_{\mathtt{pre}}, 0]$. Following a rule of thumb proposed in [19] (also cf. [20]) preaction time can be determined as $t_{\mathtt{pre}} = f_{\mathtt{pre}}/d_{\mathtt{rhp}}$ where $f_{\mathtt{pre}}$ is a selectable factor to be chosen in the interval $[5, 10]$ and $d_{\mathtt{rhp}}$ is the minimum distance of the nonminimum-phase zeros from the imaginary axis.

We choose $f_{\mathtt{pre}} = 10$ and for both inverses $t_{\mathtt{pre}} = 10$ s. The simulation results are reported in Figure 4. It shows the

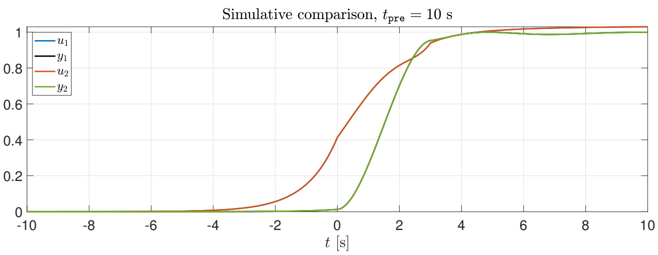

Fig. 4. Plots with $t_{\mathtt{pre}} = 10$ s. Signals u_1 and y_1 (u_2 and y_2) are plotted in blue and black (red and green) respectively. Due to the signal overlapping only the colors of u_2 and y_2 are visible.

plots of u_1 and u_2 and those of y_1 and y_2 overlapping to each other. Indeed, with a negligible error due to the inverses' preaction truncations (i.e. $\max_{t\in\mathbb{R}}|u_1(t) - u_2(t)| = 7.46 \times 10^{-4}$ and $\max_{t\in\mathbb{R}}|y_1(t) - y_2(t)| = 4.4 \times 10^{-5}$) we have $u_1 \equiv u_2$ and $y_1 \equiv y_2$. This is the expected result because the equivalence theorem (Theorem 1) predicts that the two inversion-based architectures deliver the same performances.

On the other hand, if we choose a rough truncation of the inverses $u_{P,\mathtt{d}}$ and $u_{H,\mathtt{d}}$ such as e.g. that obtained by setting $f_{\mathtt{pre}} = 2$ and consequently $t_{\mathtt{pre}} = 2$ s the simulations plotted in Figure 5 show that both u_1 and y_1 significantly differ from

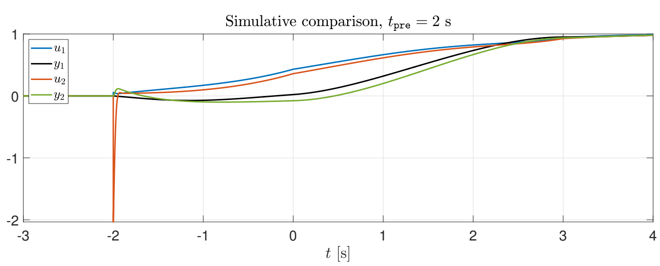

Fig. 5. Plots with $t_{\mathtt{pre}} = 2$ s. For presentation convenience the plots span the interval $[-3, 4]$ s. Signals u_1 and y_1 (u_2 and y_2) are plotted in blue and black (red and green) respectively.

u_2 and y_2 respectively ($\max_{t\in\mathbb{R}}|u_1(t) - u_2(t)| = 2.09$ and $\max_{t\in\mathbb{R}}|y_1(t) - y_2(t)| = 0.12$). Hence, when (substantially) inexact inversion is implemented the two inversion-based control architectures are no longer equivalent.

VI. CONCLUSIONS

The two main model inversion architectures, i.e. the plant and closed-loop inversion architectures, have been shown to be equivalent for the control of nonminimum-phase plants. Remarkably, this equivalence still holds when the architectures' feedback controller is nonminimum-phase too. Moreover, the equivalence is still valid in practice when a careful truncation of the preaction control is performed. However, this equivalence is lost whenever (substantially) inexact inversion is implemented such as e.g. when the preaction control is abruptly truncated (cf. simulation comparisons in Section V). Hence, notwithstanding the relevance of the presented equivalence result, architecture comparisons may be still worth to be investigated when approximate stable inverses are used (cf. [7], [8], [9], [21]).

On the other hand, with the aim to achieve high performances in control applications, future research on model inversion architectures may focus on the (feedback) controller design methodology. Indeed, apparently very few contributions have appeared in the control literature on this topic (cf. [5], [22]).

REFERENCES

[1] S. Devasia, D. Chen, and B. Paden, "Nonlinear inversion-based output tracking," *IEEE Tr. Aut. Con.*, vol. AC-41, no. 7, pp. 930–942, 1996.

[2] L. Hunt, G. Meyer, and R. Su, "Noncausal inverses for linear systems," *IEEE Tr. Aut. Con.*, vol. AC-41, pp. 608–611, 1996.

[3] S. Devasia, "Should model-based inverse inputs be used as feedforward under plant uncertainty?," *IEEE Tr. Aut. Con.*, vol. 47, pp. 1865–1871, November 2002.

[4] Y. Wu and Q. Zou, "Robust inversion-based 2-dof control design for output tracking: piezoelectric-actuator example," *IEEE Transactions on Control Systems Technology*, vol. 17, pp. 1069–1082, Sept 2009.

[5] A. Piazzi and A. Visioli, "Optimal inversion-based control for the set-point regulation of nonminimum-phase uncertain scalar systems," *IEEE Tr. Aut. Con.*, vol. 46, pp. 1654–1659, Oct. 2001.

[6] A. Piazzi and A. Visioli, "Robust set-point constrained regulation via dynamic inversion," *International Journal of Robust and Nonlinear Control*, vol. 11, pp. 1–22, January 2001.

[7] B. P. Rigney, L. Y. Pao, and D. A. Lawrence, "Model inversion architectures for settle time applications with uncertainty," in *45th IEEE Conference on Decision and Control*, pp. 6518–6524, 2006.

[8] B. P. Rigney, L. Y. Pao, and D. A. Lawrence, "Nonminimum phase dynamic inversion for settle time applications," *IEEE Transactions on Control Systems Technology*, vol. 17, no. 5, pp. 989–1005, 2009.

[9] J. Butterworth, L. Pao, and D. Abramovitch, "A comparison of control architectures for atomic force microscopes," *Asian Jou. Control*, vol. 11, pp. 175–181, March 2009.

[10] J. van Zundert and T. Oomen, "On inversion-based approches for feedforward and ilc," *Mechatronics*, vol. 50, pp. 282–291, 2018.

[11] L. Blanken and T. Oomen, "Kernel-based identification of non-causal systems with application to inverse model control," *Automatica*, vol. 114, p. 108830, 2020.

[12] J. Polderman and J. Willems, *Introduction to mathematical systems theory: a behavioral approach*. New York, NY: Springer, 1998.

[13] A. Costalunga and A. Piazzi, "A behavioral approach to inversion-based control," *Automatica*, vol. 95, pp. 433 – 445, 2018.

[14] R. Middleton, S. Graebe, A. Alén, and J. Shamma, "Design methods," in *The Control Handbook: Control Systems Fundamentals* (W. Levine, ed.), pp. **19**–1–**19**–35, CRC Press, second ed., 2011.

[15] D. Pallastrelli and A. Piazzi, "Stable dynamic inversion of nonminimum-phase scalar linear systems," in *16th IFAC World Congress on Automatic Control*, pp. 107–112, 2005.

[16] S. Skogestad and I. Postlethwaite, *Multivariable Feedback Control: analysis and design*. Wiley-Interscience, second ed., 2005.

[17] A. Piazzi and A. Visioli, "Using stable input-output inversion for minimum-time feedforward constrained regulation of scalar systems," *Automatica*, vol. 41, no. 2, pp. 305–313, 2005.

[18] A. Piazzi and A. Visioli, "Optimal noncausal set-point regulation of scalar systems," *Automatica*, vol. 37, pp. 121–127, January 2001.

[19] A. Minari, A. Piazzi, and A. Costalunga, "Polynomial interpolation for inversion-based control," *European Journal of Control*, 2020. https://doi.org/10.1016/j.ejcon.2020.01.007.

[20] H. Perez and S. Devasia, "Optimal output-transitions for linear systems," *Automatica*, vol. 39, no. 2, pp. 181–192, 2003.

[21] L. Dai, X. Li, Y. Zhu, and M. Zhang, "Quantitative analysis on tracking error under different control architectures and feedforward methods," in *2019 American Control Conference (ACC)*, pp. 5680–5686, 2019.

[22] A. Piazzi and A. Visioli, "Combining H_∞ control and dynamic inversion for robust constrained set-point regulation," *International Journal of Tomography & Statistics*, vol. 6, no. S07, pp. 63–68, 2007.