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Input-Output Jumps of Scalar Linear Systems

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Abstract: In the classic control systems analysis a known difficulty is to determine a transient output response when non-zero (discontinuous) initial conditions are present. This difficulty is overcome by means of the presented input-output jump relations. For a system with order n and relative degree $r = n - m$, these relations relate the jump discontinuities of the input and its derivatives up to the order $m - 1$ to the jump discontinuities of the output and its derivatives up to the order $n - 1$. A simplified behavior theory is used in deducing these straightforward relations. A simple, complete solution to the initial conditions problem that uses neither generalized derivatives nor ad hoc assumptions is then presented. A detailed example illustrating this solution is included.

Keywords: Linear continuous-time systems, single-input/single-output systems, input-output jump discontinuities, initial conditions, control systems education.

1. INTRODUCTION

In the classic control systems education, linear constant-coefficient differential equations and Laplace transforms are used in describing and analyzing the control system dynamics (cf. Chen (1993); Nise (2004); Dorf and Bishop (2008)). However, in explaining the transient response this classic treatment encounters difficulties or pitfalls in the case of non-zero (discontinuous) initial conditions as illustrated in Lundberg et al. (2007). When the initial conditions are known at time 0^+ the use of straightforward Laplace properties, such as e.g. the transform of the (usual) derivative function, permits to easily find the transform of the total response and then, by inverse Laplace transform, to obtain the explicit expression of the (output) response over $\mathbb{R}_{\geq 0}$. However, in the usual case the initial conditions are known at time 0^- (also called *pre-initial conditions*) and this causes a difficulty.

By using a physical reasoning, the conditions at time 0^+ (also called *post-initial conditions*) may be deduced by the pre-initial conditions in particular cases such as those of mechanical and electrical systems. An alternative, tentative approach achieving the same purpose is to obtain the post-initial conditions by the technique of impulse matching as presented with an example in Siebert (1986). But the mainstream way to determine the total response is directly use the pre-initial conditions (cf. e.g. Chen (1993)). This requires using the $\mathcal{L}_-$ definition for the Laplace transform and the generalized derivatives (cf. Kailath (1980) and Lundberg et al. (2007)). However, the use of generalized derivatives as done in this context appears somewhat unsatisfactory or convolute (cf. Grizzle (2004); Mäkilä (2006); Ahuja and Arya (2018)).

A behavioral approach (cf. Polderman and Willems (1998)) is proposed herein to determine the initial conditions at time 0^+ from those at time 0^- for continuous-time, time-invariant, scalar linear systems. This is achieved by means of a study on input-output jumps. Specifically, considering a system whose order and relative degree are n and $r = n - m$ respectively, the simplified behavior theory of Costalunga and Piazzi (2018) is used to relate the jump discontinuities of the input and its derivatives up to the order $m - 1$ to the jump discontinuities of the output and its derivatives up to the order $n - 1$ (cf. Proposition 11 and Corollary 14). Indeed, when jump discontinuities occur the differential equation associated to the system cannot be satisfied in the usual sense over the entire time axis because the input and the output cannot be differentiated up to the required orders. To overcome this obstruction, the classic mathematical notion of *weak solution* can be introduced (cf. Polderman and Willems (1998)). In particular, an appropriate integral equation replaces the differential equation (cf. (1) and (2)) so that the (simplified) *behavior* of the system can be defined as the set of all the input-output (signal) pairs that are weak solutions of the differential equation, i.e. the input-output pairs actually satisfy (in the usual sense) the corresponding integral equation over the entire time axis (cf. Definition 7).

The input-output jump relations (3)-(4) are then deduced by taking left and right limits on the kth derivative of the integral equation characterizing the weak solution at each given time instant, not only at the conventional origin time 0. These relations are also simplified in (10) by introducing a lower triangular Toeplitz matrix (defined by Markov parameters) that directly relates the output jumps to the input ones. The found relations permit to easily compute the output conditions at time 0^+ from the pre-initial conditions (involving both the input and the output at time 0^-) and the input conditions at time 0^+.

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A solution to the *initial conditions problem* (cf. Problem 17) in a behavioral setting, i.e. to determine the total response starting from an arbitrary system evolution, is also presented.

The paper is organized as follows. Section 2 presents preliminaries on $C_{\mathtt{p}}^{\infty}$, the set of piecewise C^{∞}-functions and some key notions of behavior theory such as e.g. the weak solutions and the behavior's input-output representation (cf. Theorem 10). The third section is focused on the main result, i.e. the input-output jump relations which are expressed in vector form by Proposition 11 and Corollary 14. The found relations are applied to solve the initial conditions problem (i.e. Problem 17) in Section 4. A result used in this solution is Corollary 18. It provides the initial conditions of the free response at time 0^+ from the knowledge of the pre-initial conditions. An example of solution of Problem 17 is reported in Section 5. Concluding remarks end the paper in Section 6.

Notation: The set of natural numbers is denoted by $\mathbb{N}$ (with $0 \in \mathbb{N}$). A function $f : \mathbb{R} \to \mathbb{R}$ has *continuity order* $n \in \mathbb{N}$, i.e. $f \in C^n$, if it is a continuous function with continuous derivatives up to the nth order. Function f belongs to C^{∞} if it has derivatives of all orders. Given $k \in \mathbb{N}$, the kth order derivative of f is denoted by $f^{(k)}$ or $D^k f$ with D being the derivative operator. The one-sided limits of f at a time $t \in \mathbb{R}$ are defined as $f(t^+) := \lim_{v\to t^+} f(v)$ and $f(t^-) := \lim_{v\to t^-} f(v)$; these will be referred to as right and left limit respectively. The Laplace transform of f is denoted by $\mathcal{L}[f(t)]$ or $F(s)$. The set of polynomial functions with real coefficients is denoted by $\mathcal{P}$. Given $p \in \mathcal{P}$, $deg(p)$ denotes the degree of polynomial p.

2. PRELIMINARIES AND THE SYSTEM BEHAVIOR

A behavioral description of continuous-time scalar linear systems is briefly introduced.

2.1 Preliminaries

The adopted space of signals is $C_{\mathtt{p}}^{\infty}$, the set of piecewise C^{∞}-functions. It can be presented as follows (cf. Costalunga and Piazzi (2018)). A set $S \subset \mathbb{R}$ is said to be *sparse* if for any real interval $[a, b]$, the intersection $S \cap [a, b]$ has finite cardinality or it is the empty set.

Definition 1. [$C_{\mathtt{p}}^{\infty}$, set of piecewise C^{∞}-functions] A function f belongs to $C_{\mathtt{p}}^{\infty}$, called the *set of piecewise C^{∞}-functions*, if there exists a sparse set S for which $f \in C^{\infty}(\mathbb{R}\backslash S, \mathbb{R})$ and for any $n \in \mathbb{N}$ and $t \in S$ the limits $f^{(n)}(t^-)$ and $f^{(n)}(t^+)$ exist and are finite. When f is defined in $t \in S$, conventionally $f(t) := f(t^+)$; in particular $C^{-1} := C_{\mathtt{p}}^{\infty}(\mathbb{R})$ denotes the set of piecewise C^{∞}-functions defined over the whole set of reals.

Remark 2. The behavioral approach herein presented is a simplified version of the behavior theory of Polderman and Willems (1998) that uses, as signal space, the set of locally integrable functions $\mathcal{L}_1^{\mathtt{loc}}$. Evidently $C_{\mathtt{p}}^{\infty}(\mathbb{R}) \subset \mathcal{L}_1^{\mathtt{loc}}(\mathbb{R})$, however this loss of generality appears to help in achieving applicable results in the control engineering field, especially in mechatronics (cf. Example 1 in Costalunga and Piazzi (2018)).

A useful definition on discontinuities of a function is the following.

Definition 3. [Discontinuity Sets] Given a function $f \in C_{\mathtt{p}}^{\infty}$, the following sparse sets are introduced: the zero-order discontinuity set $S_f^{(0)} := \{t \in \mathbb{R} : f(t^-) \neq f(t^+) \vee f \text{ is not defined in } t\}$; the nth-order discontinuity set $S_f^{(n)} := \{t \in \mathbb{R} : f^{(n)} \text{ does not exist in } t\}$.

The integral operator that comprises the differential operator when the exponent is negative can be introduced as follows.

Definition 4. [Integral operator] Let $f \in C_{\mathtt{p}}^{\infty}$ and define $\int f(t) \equiv \int^1 f(t) \equiv (\int f)(t) := \int_0^t f(\xi)d\xi$ and $\int^0 f := f$. Let $k \in \mathbb{Z}$, $\int^k f$ is defined by the recursion $\int^k f := \int(\int^{k-1} f)$ if $k \geq 1$ whereas $\int^k f := D^{-k} f$ if $k \leq -1$.

Lemma 5. Let $f \in C^p$ with $p \in \mathbb{N}\cup\{-1\}$ and $k \in \mathbb{N}$. Then

$$\int^k f \in C^{p+k}.$$

Lemma 6. Let $f \in C_{\mathtt{p}}^{\infty}$ and $p, k \in \mathbb{N}$. Then $D^k\big(\int^p f\big)$ is defined on $\mathbb{R}$ if $p > k$ and on $\mathbb{R} \setminus S_f^{(k-p)}$ if $p \leq k$. Furthermore

$$D^k\left(\int^p f\right) = \int^{p-k} f.$$

Proofs of the above lemmas can be found in Costalunga and Piazzi (2018).

2.2 The system behavior

Let Σ be a continuous-time scalar linear system with input $u \in C_{\mathtt{p}}^{\infty}(\mathbb{R})$ and output $y \in C_{\mathtt{p}}^{\infty}(\mathbb{R})$. Σ be defined by its transfer function

$$H(s) = \frac{b(s)}{a(s)} = \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_0}{a_n s^n + a_{n-1}s^{n-1} + \cdots + a_0}$$

with $a(s)$ and $b(s)$ being coprime polynomials having real coefficients and $a_n \neq 0$, $b_m \neq 0$, and $m \leq n$. The order and relative degree of Σ are n and $r = n - m$ respectively.

The (system) behavior of Σ (cf. Polderman and Willems (1998)) is the set of all input-output pairs that are *weak solutions* of the differential equation associated to Σ. Following Costalunga and Piazzi (2018), a pair $(u, y) \in C_{\mathtt{p}}^{\infty}(\mathbb{R})^2$ is said to be a weak solution of

$$\sum_{i=0}^{n} a_i D^i y(t) = \sum_{i=0}^{m} b_i D^i u(t) \tag{1}$$

if there exists a polynomial $g \in \mathcal{P}$ with $\deg g \leq n - 1$ for which the integral equation

$$\sum_{i=0}^{n} a_i \int^{n-i} y(t) = \sum_{i=0}^{m} b_i \int^{n-i} u(t) + g(t) \tag{2}$$

is satisfied for all $t \in \mathbb{R}$. Hence, the behavior of Σ can be formally introduced as follows.

Definition 7. [Behavior of Σ]

$$\mathcal{B} := \{\ (u, y) \in C_{\mathtt{p}}^{\infty}(\mathbb{R})^2 : (u, y) \text{ is a weak solution of (1)}\ \}.$$

A property on the continuity order of the output is the following.

Proposition 8. Let $(u, y) \in \mathcal{B}$, then $y \in C^{r-1}$.

The poles of Σ are the roots (with multiplicity) of $a(s)$. The associated concept of *pole modes* can be then introduced.

Definition 9. [Pole modes of Σ] Given a real (complex) pole $p \in \mathbb{R}$ ($p = \sigma \pm j\omega \in \mathbb{C}$) with multiplicity μ, the associated modes are

$$e^{pt}, te^{pt}, \ldots, t^{\mu-1}e^{pt} \;\; (e^{\sigma t}\cos(\omega t), e^{\sigma t}\sin(\omega t), \ldots \\ , t^{\mu-1}e^{\sigma t}\cos(\omega t), t^{\mu-1}e^{\sigma t}\sin(\omega t)).$$

All the pole modes of Σ are denoted by $m_i(t)$, $i = 1, \ldots, n$.

Denote by $h(t)$ the analytical extension over $\mathbb{R}$ of $\mathcal{L}^{-1}[H(s)]$. Then, an explicit (input-output) representation of the system behavior is the following.

Theorem 10. [Behavior's input-output representation] Define the following set

$$\mathcal{B}_{\mathtt{i/o}} := \{(u, y) \in C^{\infty}_{\mathtt{p}}(\mathbb{R})^2 : y(t) = \int_0^t h(t - v)u(v)dv \\ + \sum_{i=1}^{n} f_i m_i(t),\ t \in \mathbb{R},\ f_i \in \mathbb{R}\}.$$

Then $\mathcal{B}_{\mathtt{i/o}} = \mathcal{B}$.

The above Proposition 8 and Theorem 10 are taken from Costalunga and Piazzi (2018). (For Theorem 10 cf. the more general result provided in (Polderman and Willems, 1998, Theorem 3.3.19 at page 88).)

3. INPUT-OUTPUT JUMPS

From a causal viewpoint, jump discontinuities on the input and its derivatives cause jump discontinuities on the output and its derivatives. A set of algebraic relations between them is presented in the following result. (Previously it was presented without a formal proof in Piazzi (2004).)

Proposition 11. [Input-output jump relations] Let be given any $(u, y) \in \mathcal{B}$. Then, at any time $t \in \mathbb{R}$ the possible input and output jump discontinuities (up to the $(m-1)$th and $(n-1)$th derivative order respectively) satisfy the following relations:

$$y^{(i)}(t^+) = y^{(i)}(t^-), \quad i = 0, 1, \ldots, r-1 \quad (\text{void if } r = 0) \tag{3}$$

$$\begin{bmatrix} a_n & 0 & \ldots & 0 \\ a_{n-1} & a_n & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ a_{r+1} & \ldots & a_{n-1} & a_n \end{bmatrix} \begin{bmatrix} y^{(r)}(t^+) - y^{(r)}(t^-) \\ y^{(r+1)}(t^+) - y^{(r+1)}(t^-) \\ \vdots \\ y^{(n-1)}(t^+) - y^{(n-1)}(t^-) \end{bmatrix} = \\ \begin{bmatrix} b_m & 0 & \ldots & 0 \\ b_{m-1} & b_m & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ b_1 & \ldots & b_{m-1} & b_m \end{bmatrix} \begin{bmatrix} u(t^+) - u(t^-) \\ u^{(1)}(t^+) - u^{(1)}(t^-) \\ \vdots \\ u^{(m-1)}(t^+) - u^{(m-1)}(t^-) \end{bmatrix} \\ (\text{void if } m = 0). \tag{4}$$

Proof. Consider any pair $(u, y) \in \mathcal{B}$ and choose any time $t \in \mathbb{R}$. Then, relations (3) derive straightforwardly from Proposition 8. (The output has always continuity order $r - 1$.) Pair (u, y) satisfies the integral equation (2) which can be written as

$$\sum_{i=0}^{n} a_i \int^{n-i} y(v) = \sum_{i=0}^{n} b_i \int^{n-i} u(v) + g(v), \quad v \in \mathbb{R} \tag{5}$$

having set $b_i = 0$, $i = m+1, \ldots, n$ for the case $r \geq 1$.

Consider any $k \in \mathbb{N}$, $0 \leq k \leq n-1$ and take the kth derivative of (5). By virtue of Lemma 6 there exists a neighborhood of t, $I_k(t)$, for which this derivative can be expressed as

$$\sum_{i=0}^{n} a_i \int^{n-i-k} y(v) = \sum_{i=0}^{n} b_i \int^{n-i-k} u(v) + D^k g(v), \\ v \in I_k(t) \setminus \{t\}. \tag{6}$$

By some algebraic manipulations and by taking into account that an integral operator having a negative exponent is actually a derivative operator (cf. Definition 4), (6) is written as

$$\sum_{i=0}^{k} a_{n-i} D^{k-i} y(v) - \sum_{i=0}^{k} b_{n-i} D^{k-i} u(v) = - \sum_{i=0}^{n-(k+1)} a_i \\ \int^{n-i-k} y(v) + \sum_{i=0}^{n-(k+1)} b_i \int^{n-i-k} u(v) + D^k g(v), \\ v \in I_k(t) \setminus \{t\}. \tag{7}$$

By Lemma 5, the right-hand side of (7) is a sum of continuous functions. Hence, take the right and left limits of (7) at t to obtain

$$\begin{cases} \sum_{i=0}^{k} a_{n-i} D^{k-i} y(t^+) - \sum_{i=0}^{k} b_{n-i} D^{k-i} u(t^+) = c \\ \sum_{i=0}^{k} a_{n-i} D^{k-i} y(t^-) - \sum_{i=0}^{k} b_{n-i} D^{k-i} u(t^-) = c \end{cases} \tag{8}$$

with $c \in \mathbb{R}$,

$$c = - \sum_{i=0}^{n-(k+1)} a_i \int^{n-i-k} y(t) + \sum_{i=0}^{n-(k+1)} b_i \int^{n-i-k} u(t) \\ + D^k g(t).$$

Form the difference of the equations in (8) we eventually have

$$\sum_{i=0}^{k} a_{n-i} \Big(D^{k-i} y(t^+) - D^{k-i} y(t^-) \Big) = \\ \sum_{i=0}^{k} b_{n-i} \Big(D^{k-i} u(t^+) - D^{k-i} u(t^-) \Big), \quad 0 \leq k \leq n-1. \tag{9}$$

When $r = 0$, note that the above relations (9) are the scalar version of the vector relation in (4).

Now, consider the case $r \geq 1$. Relations (9) are still valid, in particular on the index subset $r \leq k \leq n-1$. By taking into account that $D^i y(t^+) - D^i y(t^-) = 0$ and $b_{n-i} = 0$, $i = 0, \ldots, r-1$ (cf. (3) and the assumption on (5) respectively), these relations can be simplified as follows:

$$\sum_{i=0}^{k-r} a_{n-i} \Big(D^{k-i} y(t^+) - D^{k-i} y(t^-) \Big) = \\ \sum_{i=r}^{k} b_{n-i} \Big(D^{k-i} u(t^+) - D^{k-i} u(t^-) \Big), \ r \leq k \leq n-1.$$

The above scalar relations can be then rewritten in vector form to obtain the input-output jump relations (4) and this concludes the proof. □

Remark 12. The set of relations (3) and the vector equation (4) form a set of n scalar linear relations. When $r = 0$ the set (3) is empty whereas when $m = 0$ equation (4) is absent.

Let us introduce the Markov parameters of Σ, h_i, $i \in \mathbb{N}$ for which $H(s) = \sum_{i=0}^{\infty} h_i s^{-i}$ (cf. Chen (1998)). The next result is a useful technical lemma. (Without loss of generality we assume $a_n = 1$ in the following.)

Lemma 13. The first $n+1$ Markov parameters of Σ can be recursively obtained by means of the following relations:

$$h_i = 0, \quad 0 \leq i \leq r-1, \quad \text{(void if } r = 0\text{)};$$
$$\begin{cases} h_r = b_m \\ h_{r+i} = b_{m-i} - \sum_{j=1}^{i} a_{n-j} h_{r+i-j}, \quad 1 \leq i \leq m \end{cases}.$$

For brevity a proof is omitted. (It can be found in Chen (1998) when $r = 1$.)

Corollary 14. Let be given any (u, y) in $\mathcal{B}$ and any time $t \in \mathbb{R}$. Then, the input-output jump relations (4) can also be expressed as

$$\begin{bmatrix} y^{(r)}(t^+) - y^{(r)}(t^-) \\ y^{(r+1)}(t^+) - y^{(r+1)}(t^-) \\ \vdots \\ y^{(n-1)}(t^+) - y^{(n-1))}(t^-) \end{bmatrix} = \begin{bmatrix} h_r & 0 & \dots & 0 \\ h_{r+1} & h_r & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ h_{n-1} & \dots & h_{r+1} & h_r \end{bmatrix} \begin{bmatrix} u(t^+) - u(t^-) \\ u^{(1)}(t^+) - u^{(1)}(t^-) \\ \vdots \\ u^{(m-1)}(t^+) - u^{(m-1)}(t^-) \end{bmatrix}. \tag{10}$$

Proof of the above corollary is omitted for brevity.

Remark 15. It is worth stressing that the input-output jump relations presented in Proposition 11 and Corollary 14 hold at any instant of the time axis not only at the origin time 0. From this viewpoint, the time 0 has nothing special. Indeed, jump discontinuities in the input-output evolution can happen at any time instant for which the found relations still hold. Moreover, if at a given time t the input has not discontinuities up to the derivative of order $m-1$, i.e. $u^{(i)}(t^-) = u^{(i)}(t^+)$, $i = 0, 1, \dots, m-1$ then by the input-output relations (3)-(4) it follows in turn that the output too has no discontinuities up to the derivative of order $n-1$, i.e. $y^{(i)}(t^-) = y^{(i)}(t^+)$, $i = 0, 1, \dots, n-1$. When this happen for some neighborhood N of t the input $u \in C^{m-1}(N)$ and $y \in C^{n-1}(N)$ (cf. Proposition 4 in Costalunga and Piazzi (2018)).

Remark 16. Note that relations (10) may be alternatively deduced from a state-space approach by expressing the Markov parameters h_i as $cA^{i-1}b$, $i = 1, 2, \dots$ in the case of $r \geq 1$ and state-space model (A, b, c) (cf. Kailath (1980)).

4. APPLICATION TO THE INITIAL CONDITIONS PROBLEM

As an application of the found input-output jump relations (3)-(4) the following initial conditions problem is considered.

Problem 17. [The initial conditions problem] Let be given any input-output pair $(u_0, y_0) \in \mathcal{B}$ whose signals u_0, y_0 are known for $t < 0$ and suppose that, at time $t = 0$, a new input $u_1(t)$, $t \geq 0$ is applied to Σ. Find the corresponding output $y_1(t)$, $t \geq 0$.

Solution to this problem is proposed as follows. Let the signals $u_0|u_1$, $y_0|y_1$ be defined as

$$u_0|u_1(t) := \begin{cases} u_0(t) \text{ if } t < 0 \\ u_1(t) \text{ if } t \geq 0 \end{cases}, \; y_0|y_1(t) := \begin{cases} y_0(t) \text{ if } t < 0 \\ y_1(t) \text{ if } t \geq 0 \end{cases}$$

so that evidently $(u_0|u_1, y_0|y_1) \in \mathcal{B}$. By Theorem 10 there exist real coefficients f_i, $i = 1, \dots, n$ for which

$$y_0|y_1(t) = \int_0^t h(t-v) u_0|u_1(v) dv + \sum_{i=1}^{n} f_i m_i(t), \quad t \in \mathbb{R}. \tag{11}$$

Define

$$y_{1\mathtt{d}}(t) := \int_0^t h(t-v) u_1(v) dv, \; t \geq 0 \tag{12}$$

and

$$y_{1\mathtt{e}}(t) := \sum_{i=1}^{n} f_i m_i(t), \; t \in \mathbb{R} \tag{13}$$

so that (11) implies

$$y_1(t) = y_{1\mathtt{d}}(t) + y_{1\mathtt{e}}(t), \; t \geq 0, \tag{14}$$

i.e. the *total response* of Σ is the sum of the *forced response* $y_{1\mathtt{d}}(t)$ and the *free* (or *natural*) *response* $y_{1\mathtt{e}}(t)$ (cf. Levine (2010)). The forced response is then determined by the convolution of $h(t)$ and $u_1(t)$ (or equivalently by $\mathcal{L}^{-1}[H(s)U_1(s)]$ with $U_1(s) := \mathcal{L}[u_1(t)]$) whereas the free response can be determined by means of the input-output jump relations (3)-(4). Indeed, relation (11) holds for any $u_1 \in C^{\infty}_{\mathtt{p}}(\mathbb{R}_{\geq 0})$, hence if $u_1(t) = 0$, $t \geq 0$ it follows that $(u_0|0, y_0|y_{1\mathtt{e}}) \in \mathcal{B}$. By applying Proposition 11 and Corollary 14 to pair $(u_0|0, y_0|y_{1\mathtt{e}})$ the following straightforward result is obtained.

Corollary 18. Let us consider the assumptions of Problem 17. Then, the free response $y_{1\mathtt{e}}(t)$ (13) has initial conditions at time 0^+ given by the following relations:

$$y_{1\mathtt{e}}^{(i)}(0^+) = y_0^{(i)}(0^-), \; i = 0, 1, \dots, r-1 \quad \text{(void if } r = 0\text{)} \tag{15}$$

$$\begin{bmatrix} y_{1\mathtt{e}}^{(r)}(0^+) \\ y_{1\mathtt{e}}^{(r+1)}(0^+) \\ \vdots \\ y_{1\mathtt{e}}^{(n-1)}(0^+) \end{bmatrix} = \begin{bmatrix} y_0^{(r)}(0^-) \\ y_0^{(r+1)}(0^-) \\ \vdots \\ y_0^{(n-1)}(0^-) \end{bmatrix} - \begin{bmatrix} h_r & 0 & \dots & 0 \\ h_{r+1} & h_r & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ h_{n-1} & \dots & h_{r+1} & h_r \end{bmatrix} \begin{bmatrix} u_0(0^-) \\ u_0^{(1)}(0^-) \\ \vdots \\ u_0^{(m-1)}(0^-) \end{bmatrix}$$

$$\text{(void if } m = 0\text{)} \tag{16}$$

Proof. Relations (15) are just those in (3) applied to the output $y_0|y_{1e}$ at time 0. On the other hand, the vector equality (16) follows from that in (10) by taking into account that the initial conditions of the input $u_0|0$ at time 0^+ are all zeros. □

A way to determine the free response is to compute the coefficients f_i's appearing in (13). Take the derivatives of $y_{1e}(t)$ up to the order $n-1$ and evaluate them at time 0^+ to obtain the following algebraic linear equation in the unknowns f_i's:

$$\begin{bmatrix} m_1(0^+) & \dots & m_n(0^+) \\ m_1^{(1)}(0^+) & \dots & m_n^{(1)}(0^+) \\ \vdots & & \vdots \\ m_1^{(n-1)}(0^+) & \dots & m_n^{(n-1)}(0^+) \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix} = \begin{bmatrix} y_{1e}(0^+) \\ y_{1e}^{(1)}(0^+) \\ \vdots \\ y_{1e}^{(n-1)}(0^+) \end{bmatrix}. \tag{17}$$

The right-hand side of (17) is computed by means of Corollary 18. Then, the f_i's can be uniquely determined because the coefficient matrix in (17), denoted by M in the following, is always nonsingular. For example, if all the poles are simple and real (i.e. the roots of $a(s)$ are $p_i \in \mathbb{R}$, $i = 1, \dots, n$ and $p_i \neq p_j$ if $i \neq j$) M becomes the classic Vandermonde matrix

$$\begin{bmatrix} 1 & 1 & \dots & 1 \\ p_1 & p_2 & \dots & p_n \\ \vdots & \vdots & & \vdots \\ p_1^{n-1} & p_2^{n-1} & \dots & p_n^{n-1} \end{bmatrix}$$

whose determinant is $\prod_{1\le i<j\le n}(p_i - p_j) \neq 0$. When there are poles (real or complex) with multiplicities, M becomes a generalized Vandermonde matrix which is still nonsingular (cf. Kalman (1984)).

Another way to find the free response (13) is the Laplace transform method. The pair $(u_0|0, y_0|y_{1e})$ satisfies the integral equation (2). Take the the nth derivative of this integral equation on $[0, +\infty)$ to obtain

$$\sum_{i=0}^{n} a_i D^i y_{1e}(t) = 0$$

and by applying the Laplace transform

$$\sum_{i=0}^{n} a_i s^i Y_{1e}(s) - \sum_{i=1}^{n} a_i \sum_{j=0}^{i-1} y_{1e}^{(i-1-j)}(0^+) s^j = 0$$

and eventually

$$Y_{1e}(s) = \frac{\sum_{i=1}^{n}\sum_{j=0}^{i-1} a_i y_{1e}^{(i-1-j)}(0^+) s^j}{a(s)}. \tag{18}$$

The free response is then given by

$$y_{1e}(t) = \mathcal{L}^{-1}[Y_{1e}(s)].$$

The explicit computation of this inverse Laplace transform can be routinely performed by partial fraction decomposition and subsequent application of Laplace table correspondences.

5. AN EXAMPLE

Consider a system Σ having transfer function (cf. Piazzi and Visioli (2005))

$$H(s) = -4\,\frac{(s-1)(s+1)}{(s+2)(s^2+s+2)}.$$

Its order and relative degree are $n = 3$ and $r = 1$ respectively. The following instance of the *initial conditions problem* is set (cf. Problem 17): Let $u_0(t) = sin(t)$ and $y_0(t) = 4\sqrt{\frac{2}{5}}sin\big(t - atan\,(3)\big)$, $t \in \mathbb{R}$ for which $(u_0, y_0) \in \mathcal{B}$ and suppose at time $t = 0$ a new input $u_1(t) = 1$, $t \geq 0$ is applied. Find the corresponding output $y_1(t)$, $t \geq 0$.

Solution to this problem can be given as follows (cf. Section 4). Output y_1 is total response for which $y_1(t) = y_{1d}(t) + y_{1e}(t)$, $t \geq 0$ (cf. (14)). The computation of the forced response y_{1d} does not involve the initial conditions. In a customary way, it can be done by means of the convolution integral (12) or by a Laplace procedure. With the latter $y_{1d}(t) = \mathcal{L}^{-1}[H(s)U_1(s)]$ with $U_1(s) = s^{-1}$ so that by partial fraction decomposition and inverse Laplace transform:

$$y_{1d}(t) = 1 + \frac{3}{2}e^{-2t} - \frac{5}{2}e^{-\frac{1}{2}t}\cos(\frac{\sqrt{7}}{2}t) - \frac{9\sqrt{7}}{14}e^{-\frac{1}{2}t}\sin(\frac{\sqrt{7}}{2}t). \tag{19}$$

The free response is $y_{1e}(t) = f_1 m_1(t) + f_2 m_2(t) + f_3 m_3(t)$ (cf. (13) and Definition 9 on pole modes of Σ) and can be determined by means of Corollary 18. The pre-initial conditions, i.e. the initial conditions of signals u_0 and y_0 at 0^- are the following: $u_0(0^-) = 0$, $u_0^{(1)}(0^-) = 1$ and $y_0(0^-) = -\frac{12}{5}$, $y_0^{(1)}(0^-) = \frac{4}{5}$, $y_0^{(2)}(0^-) = \frac{12}{5}$. The transfer function can be expressed as $H(s) = \frac{-4s^2+4}{s^3+3s^2+4s+4}$ and by applying Lemma 13 the Markov parameters $h_0 = 0$, $h_1 = -4$, and $h_2 = 12$ are determined. Therefore, from (15) and (16) we obtain the free response initial conditions at 0^+:

$$y_{1e}(0^+) = y_0(0^-) = -\frac{12}{5}, \tag{20}$$

and

$$\begin{bmatrix} y_{1e}^{(1)}(0^+) \\ y_{1e}^{(2)}(0^+) \end{bmatrix} = \begin{bmatrix} \frac{4}{5} \\ \frac{12}{5} \end{bmatrix} - \begin{bmatrix} -4 & 0 \\ 12 & -4 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{4}{5} \\ \frac{32}{5} \end{bmatrix}. \tag{21}$$

The poles of Σ are -2 and $-\frac{1}{2} \pm j\frac{\sqrt{7}}{2}$ with associated modes $m_1(t) = e^{-2t}$, $m_2(t) = e^{-\frac{1}{2}t}\cos(\frac{\sqrt{7}}{2}t)$, and $m_3(t) = e^{-\frac{1}{2}t}\sin(\frac{\sqrt{7}}{2}t)$. The modes are used in defining the Vandermonde matrix M in (17) so as to compute the coefficients f_i's of the free response:

$$\begin{bmatrix} 1 & 1 & 0 \\ -2 & -\frac{1}{2} & \frac{\sqrt{7}}{2} \\ 4 & -\frac{3}{2} & -\frac{\sqrt{7}}{2} \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} -\frac{12}{5} \\ \frac{4}{5} \\ \frac{32}{5} \end{bmatrix} \implies \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} \frac{3}{5} \\ -3 \\ \frac{1}{\sqrt{7}} \end{bmatrix};$$

hence,

$$y_{1e}(t) = \frac{3}{5}e^{-2t} - 3e^{-\frac{1}{2}t}\cos\big(\frac{\sqrt{7}}{2}t\big) + \frac{1}{\sqrt{7}}e^{-\frac{1}{2}t}\sin\big(\frac{\sqrt{7}}{2}t\big). \tag{22}$$

Alternatively, still using the found initial conditions at 0^+ in (20)-(21), the free response (22) can be directly determined (cf. (18)) by the inverse Laplace transform of

$$Y_{1e}(s) = \frac{-\frac{12}{5}s^2 - \frac{32}{5}s - \frac{4}{5}}{s^3 + 3s^2 + 4s + 4}.$$

Eventually, the sum of the forced and free responses (19) and (22) gives the sought total response ($t \geq 0$):

$$y_1(t) = 1 + \frac{21}{10}e^{-2t} - \frac{11}{2}e^{-\frac{1}{2}t}\cos(\frac{\sqrt{7}}{2}t) - \frac{\sqrt{7}}{2}e^{-\frac{1}{2}t}\sin(\frac{\sqrt{7}}{2}t).$$

The following figures illustrate all the involved signals. Figure 1 plots the input $u_0|u_1(t)$. The forced and free responses are plotted in Figure 2 and Figure 3 plots the resulting output $y_0|y_1(t)$.

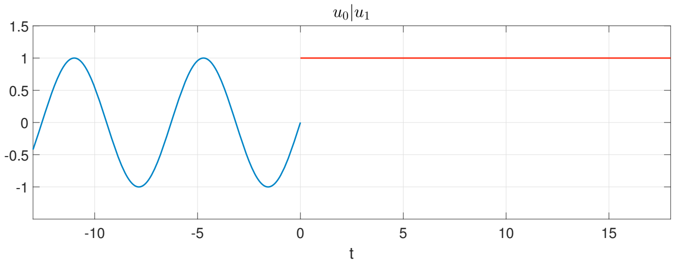

Fig. 1. Plot of the input signal $u_0|u_1$ in the time interval $[-13, 18]$. The blue and red curves are the plots of $u_0(t) = \sin(t)$, $t \in [-13, 0)$ and $u_1(t) = 1$, $t \in [0, 18]$ respectively.

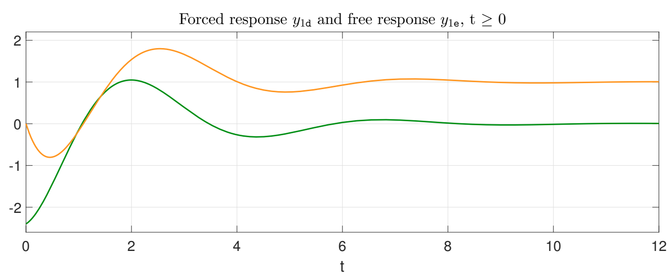

Fig. 2. Plot of the forced response y_{1d} (in orange) and free response y_{1e} (in green). The sum of these signals yields y_1.

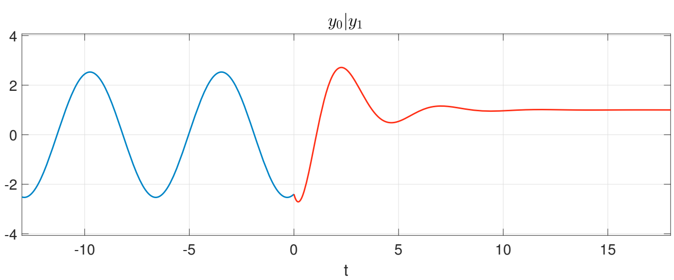

Fig. 3. Plot of the output signal $y_0|y_1$ in the time interval $[-13, 18]$. The blue curve is $y_0(t)$, $t \in [-13, 0)$ whereas the red one is $y_1(t)$, $t \in [0, 18]$.

6. CONCLUSIONS

The jump discontinuities of the input vector $(u(t), u^{(1)}(t), \ldots, u^{(m-1)}(t))$ cause jump discontinuities on the output vector $(y(t), y^{(1)}(t), \ldots, y^{(n-1)}(t))$ at any given time $t \in \mathbb{R}$. Straightforward algebraic relations between them have been established in (3) and (4) (or (10)) for linear time-invariant scalar systems. Significantly, these relations have been obtained by means of a simplified behavioral approach that avoids generalized derivatives and ad hoc assumptions. With ease, the found input-output jump relations have direct application to solve the initial conditions problem. The present findings may be a useful complement in the classic control systems education. A possible extension to MIMO (multiple-input multiple-output) systems may be pursued. In this case the *input-output decoupling* should be a relevant issue of this extension (cf. Kavaja et al. (2018)).

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