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A behavioral approach to inversion-based control ☆

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ABSTRACT

A new simplified behavior theory is proposed to address inversion-based control for linear, nonminimum-phase SISO systems. The chosen space of signals is the set of piecewise C^∞-functions and input–output pairs (as weak solutions) satisfy a differential–integral equation with additional smoothness requirements. A related key result is the output–input (or inverse) representation of the behavior set that leads to the solution of a general stable inversion problem where polynomially unbounded, noncausal desired outputs are allowed. It is shown that this problem has a solution if and only if the *smoothness degree* of the desired output is greater than or equal to the system relative degree minus one. When this straightforward condition is satisfied, a closed-form expression provides the inverse input. Then, an analysis on *preaction* and *postaction* control follows. Two examples are included showing the relevance of output signal design in control applications.

1. Introduction

Feedforward control methods have recently attracted significant interest in the technical literature (Singh, 2010). Indeed, feedforward can effectively complement feedback control and in such a way it helps to improve the performances in the control and regulation of systems and processes. Among various methods (bang–bang control, input shaping, etc.), inversion-based (feedforward) control has found its way in mid 90s and subsequent years (Devasia, Chen, & Paden, 1996; Hunt, Meyer, & Su, 1996; Piazzi & Visioli, 2001b; Zoua & Devasia, 1999). The basic idea of this feedforward method is first to design the output signal according to the specific application. Then, by system inversion, the (inverse) input that causes the desired output is determined.

In applying the method to nonminimum-phase systems, i.e. systems with an unstable zero dynamics (Isidori, 1995), a difficulty emerged. Indeed, the standard inversion leads to an unbounded input due to the fact that, for these systems, the inverse system is unstable. This theoretical obstruction was overcome by the works in Bayo (1987), Chen (1993), Devasia et al. (1996) and Hunt et al. (1996). The idea leading to the breakthrough was to search for solutions among noncausal signals. For nonlinear (input) affine systems with hyperbolic zero dynamics, Devasia et al. (1996) proposed a solution based on a (noncausal) stable inversion procedure. The nonlinear system is transformed into normal form and a bounded noncausal solution of the zero dynamics is found by solving a fixed-point operator equation by means of Picard iterations. In such a way, a corresponding bounded noncausal input is finally determined. An analogous solution in the linear case was proposed by Hunt et al. (1996). Their procedure was easier to apply also because the bounded solution of the zero dynamics is directly given, without any iteration, by a convolution integral. Various extensions appeared in Devasia (1997, 1999) and Devasia and Paden (1998). In Devasia and Paden (1998), stable inversion was established for time-varying nonlinear systems. In the case of nonhyperbolic zero dynamics, procedures of approximate stable inversion were proposed in Devasia (1997, 1999) for linear and nonlinear systems respectively. In all these cited works, stable inversion was established in a state-space setting with the exception of the seminal work (Bayo, 1987) that used a finite-element frequency-domain formulation. For linear SISO (single-input single output) systems, stable inversion was also deduced by using a transfer function technique (based on a study of the unstable causal inverse) in Pallastrelli and Piazzi (2005) and the two-sided Laplace transform in Sogo (2010).

For linear, nonminimum-phase SISO systems, this paper proposes a new approach to the inversion-based control using behavior theory. A main motivation of this approach is to provide for the control applications an easy-to-apply stable inversion procedure based on the classic concepts of system transfer function and its inverse. To achieve this a revisiting of behavior theory is necessary. More specifically, two kinds of new results are presented. The first is a simplified version of Polderman and Willems's

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behavior theory (Polderman & Willems, 1998). Indeed, the chosen space of signals is C_p^∞, i.e. the set of piecewise C^∞-functions (cf. Definition 2), instead of the more general $\mathcal{L}_1^{loc}$ of the classic theory. This choice – which is not restrictive and fully justified for the needs of inversion-based control – permits to obtain with ease the basic results of behavior theory as well as new results that are crucial in developing inversion-based control. Among these we find the characterization of *weak solutions* as those satisfying a differential–integral equation with additional smoothness conditions (Theorem 3) and the output–input (or inverse) representation of the behavior set (Theorem 5). The second is the inversion-based control deduced from the presented behavior theory. The main result here is the formulation and solution of a general stable inversion problem that admits polynomially unbounded, noncausal desired outputs. This problem has a solution (Theorem 6) if and only if the *smoothness degree* (cf. Definition 9) of the desired output is greater than or equal to the system relative degree minus one. When this straightforward condition is satisfied the inverse input is provided by a closed-form expression that extends an analogous expression proposed in Pallastrelli and Piazzi (2005). A few parts of this paper were presented in Costalunga and Piazzi (2015), specifically those regarding the smoothness degree, the polynomial smoothing scheme (46), the relative degree condition (38) of Theorem 6 (given in Costalunga and Piazzi (2015) without proof and in the restricted case of causal desired outputs), and Example 2 in Section 5 (with minor modifications).

The paper is organized as follows. Section 2 presents preliminary definitions and results on C_p^∞, discontinuity sets and the algebra of constant-coefficient (integral or differential–integral) operators. The behavior theory is presented in Section 3 where a preamble is followed by two parts devoted to behavior's properties (Section 3.1) and behavior's representations (Section 3.2). The stable inversion problem (Problem 1) and its solution (Theorem 6) are reported in Section 4 along with results on *preaction* and *postaction* control (Propositions 7 and 8 respectively). Inversion-based control is illustrated by two examples and a discussion in Section 5. Concluding remarks end the paper in Section 6.

Notation: The set of natural and positive natural numbers are denoted by $\mathbb{N}$ and $\mathbb{N}^+$ respectively. We say that a real function $f : \mathbb{R} \to \mathbb{R}$ has *continuity order* n if it belongs to C^n, the set of continuous functions with continuous derivatives till the nth-order. It belongs to C^∞ when f has derivatives of any order. The nth-order derivative of a real function f is denoted by $f^{(n)}$ or $D^n f$.

A function f is said to be *causal* if $f(t) = 0$ for all $t < 0$. The Heaviside function is denoted by $1(t)$ according to: $1(t) := 1$ if $t \geq 0$ and $1(t) := 0$ if $t < 0$.

The Laplace transform and the inverse Laplace transform are denoted by $\mathcal{L}[\cdot]$ and $\mathcal{L}^{-1}[\cdot]$ respectively.

The set of polynomial with real (complex) coefficients is denoted by $\mathcal{P}$. The degree of $p \in \mathcal{P}$ is $\deg p$. If p is the null polynomial then $\deg p = -1$ conventionally. The set of Laurent polynomials is denoted by $\mathcal{P}_L := \{p : \mathbb{C} \to \mathbb{C} \mid p = \sum_{i=-m}^{n} p_i s^i,\ n, m \in \mathbb{N},\ p_i \in \mathbb{R}(\mathbb{C})\}$.

2. Preliminaries

Given a real function f and $n \in \mathbb{N}$, the following shorthand notation for left and right limits will be used in the paper:

$$f^{(n)}(t^-) := \lim_{v \to t^-} f^{(n)}(v),\ f^{(n)}(t^+) := \lim_{v \to t^+} f^{(n)}(v).$$

Definition 1 (*Sparse Sets*)**.** A set $S \subset \mathbb{R}$ is said to be *sparse* if for any real interval $[a, b]$, $S \cap [a, b]$ has finite cardinality or it is the empty set.

Definition 2 (C_p^∞, *Set of Piecewise C^∞-functions*)**.** A function f belongs to C_p^∞, called the *set of piecewise C^∞-functions*, if there exists a sparse set S for which $f \in C^\infty(\mathbb{R} \setminus S, \mathbb{R})$ and for any $n \in \mathbb{N}$ and $t \in S$ the limits $f^{(n)}(t^-)$ and $f^{(n)}(t^+)$ exist and are finite.

When f is defined in $t \in S$, conventionally $f(t) := f(t^+)$; in particular $C^{-1} := C_p^\infty(\mathbb{R})$ denotes the set of piecewise C^∞-functions defined over the whole set of reals.

Remark 1. Note that given $f \in C_p^\infty$, $n \in \mathbb{N}$ the limits $f^{(n)}(t^-)$, $f^{(n)}(t^+)$ exist and are finite for any $t \in \mathbb{R}$. Also remark that if $f \in C_p^\infty(\mathbb{R})$ then $f(t) = f(t^+)$ for any $t \in \mathbb{R}$, i.e. f is right continuous over $\mathbb{R}$.

Definition 3 (*Discontinuity Sets*)**.** Given a function $f \in C_p^\infty$ and $n \in \mathbb{N}^+$, the following sparse sets are introduced: the zero-order discontinuity set $S_f^{(0)} := \{t \in \mathbb{R} : f(t^-) \neq f(t^+) \vee f$ is not defined in $t\}$; the nth-order discontinuity set $S_f^{(n)} := \{t \in \mathbb{R} : f^{(n)}$ does not exist in $t\}$; the discontinuity set (of any order) $S_f := \{t \in \mathbb{R} : \exists p \in \mathbb{N} \ni t \in S_f^{(p)}\}$.

It is worth noting that C_p^∞ is closed under derivation and integration. Indeed, if $f \in C_p^\infty$, then $Df \in C_p^\infty$ and $\int_a^t f(\xi)d\xi \in C_p^\infty$, $a \in \mathbb{R}$. For example, consider the Heaviside function: $1(t) \in C_p^\infty(\mathbb{R}) \subset C_p^\infty$ and $S_{1(t)}^{(0)} = S_{1(t)} = \{0\}$. The derivative is $D1(t) \in C_p^\infty$ with $S_{D1(t)}^{(0)} = S_{D1(t)} = \{0\}$; the integral is $f(t) := \int_a^t 1(\xi)d\xi \in C_p^\infty(\mathbb{R}) \subset C_p^\infty$ with $S_f^{(0)} = \emptyset$ and $S_f^{(1)} = S_f = \{0\}$. Focusing on integration, it is useful to introduce the integral operator as follows.

Definition 4 (*Integral Operator*)**.** Let $f \in C_p^\infty$, define $\int f(t) \equiv \int^1 f(t) \equiv (\int f)(t) := \int_0^t f(\xi)d\xi$ and $\int^0 f := f$. Consider $k \in \mathbb{Z}$, $\int^k f$ is defined by the recursion $\int^k f := \int(\int^{k-1} f)$ if $k \geq 1$ whereas $\int^k f := D^{-k} f$ if $k \leq -1$.

Remark 2. The integral operator defined as above also comprises the differential operator when the exponent is negative.

Lemma 1. *Let $f \in C^p$ with $p \geq -1$ and $k \in \mathbb{N}$. Then*

$$\int^k f \in C^{p+k}. \qquad (1)$$

Lemma 2. *Let $f \in C_p^\infty$ and $p, k \in \mathbb{N}$. Then $D^k(\int^p f)$ is defined on $\mathbb{R}$ if $p > k$ and on $\mathbb{R} \setminus S_f^{(k-p)}$ if $p \leq k$. Moreover*

$$D^k\left(\int^p f\right) = \int^{p-k} f.$$

Lemma 3. *Let $f \in C_p^\infty \cap C^0$ then*

$$\int Df(t) = f(t) - f(0),\ t \in \mathbb{R}. \qquad (2)$$

Remark 3. The assumptions of Lemma 3 do not require that f be differentiable. Hence, relation (2) is a slight generalization of the fundamental theorem of Calculus (Rudin, 1987).

A useful generalization of Lemma 3 is the following.

Lemma 4. *Let $f \in C_p^\infty \cap C^{p-1}$ with $p \in \mathbb{N}^+$ and consider $q \in \mathbb{N}^+$ such that $q \geq p$. Then*

$$\int^q D^p f(t) = \int^{q-p} f(t) - \sum_{i=0}^{p-1} \frac{f^{(i)}(0)}{(i+q-p)!} t^{i+q-p},\ t \in \mathbb{R}. \qquad (3)$$

2.1. Algebra of constant-coefficient operators

The following sets of constant-coefficient differential and integral operators can be introduced:

$$\mathcal{S}_{\mathrm{d}} := \left\{ A : C_{\mathrm{p}}^{\infty} \to C_{\mathrm{p}}^{\infty} \,|\, A = \sum_{i=0}^{n} a_i D^i,\, n \in \mathbb{N},\, a_i \in \mathbb{R}(\mathbb{C}) \right\},$$

$$\mathcal{S}_{\mathrm{i}} := \left\{ A : C_{\mathrm{p}}^{\infty} \to C_{\mathrm{p}}^{\infty} \,|\, A = \sum_{i=0}^{n} a_i \int^{i},\, n \in \mathbb{N},\, a_i \in \mathbb{R}(\mathbb{C}) \right\}.$$

Addition and multiplication by scalar of constant coefficient operators can be defined in the usual way and, as a consequence, $\mathcal{S}_{\mathrm{d}}$ and $\mathcal{S}_{\mathrm{i}}$ can be seen as vector spaces. Composition of A and B, both belonging to $\mathcal{S}_{\mathrm{d}}$ (or $\mathcal{S}_{\mathrm{i}}$) can be introduced as $A \circ B$ or simply as AB. Then $AB \in \mathcal{S}_{\mathrm{d}}$ (or $\mathcal{S}_{\mathrm{i}}$) and moreover $AB = BA$, i.e. constant-coefficient operators commute.

Given the operator $A = \sum_{i=0}^{n} a_i D^i$ (or $A = \sum_{i=0}^{n} a_i \int^{i}$), the polynomial associated to A, denoted by $p_A \in \mathcal{P}$, is defined as

$$p_A(s) := \sum_{i=0}^{n} a_i s^i. \tag{4}$$

On the other hand, given a polynomial $a \in \mathcal{P}$ with $a(s) = \sum_{i=0}^{n} a_i s^i$, the associated constant-coefficient differential and integral operators can be defined as

$$a(D) := \sum_{i=0}^{n} a_i D^i, \quad a(\textstyle\int) := \displaystyle\sum_{i=0}^{n} a_i \int^{i}.$$

These definitions establish a one-to-one correspondence between $\mathcal{P}$ and $\mathcal{S}_{\mathrm{d}}$ (or $\mathcal{S}_{\mathrm{i}}$). Then, straightforward identities follow: given $A \in \mathcal{S}_{\mathrm{d}}$ (or $\mathcal{S}_{\mathrm{i}}$) then $p_A(D) = A$ (or $p_A(\int) = A$); given $a \in \mathcal{P}$ then $p_{a(D)} = a$ and $p_{a(\int)} = a$.

For differential operators only, the following result was presented in Apostol (1969). Herein, it is extended to comprise both differential and integral operators.

Theorem 1. *Let $A, B \in \mathcal{S}_{\mathrm{d}}$ (or $\mathcal{S}_{\mathrm{i}}$), p_A and p_B be the associated polynomials, and $\lambda \in \mathbb{R}(\mathbb{C})$. Then*

(a) *$A = B$ if and only if $p_A = p_B$,*
(b) *$p_{A+B} = p_A + p_B$,*
(c) *$p_{\lambda A} = \lambda \cdot p_A$,*
(d) *$p_{AB} = p_A \cdot p_B$.*

As a consequence of Theorem 1, any sum decomposition or factorization of a polynomial determines a corresponding sum decomposition or factorization of the associated differential or integral operator and vice versa.

The set of constant-coefficient differential–integral operators is also introduced:

$$\mathcal{S}_{\mathrm{d\text{-}i}} := \left\{ A : C_{\mathrm{p}}^{\infty} \to C_{\mathrm{p}}^{\infty} \,|\, A = \sum_{i=-m}^{n} a_i \int^{i},\, m, n \in \mathbb{N},\, a_i \in \mathbb{R}(\mathbb{C}) \right\}.$$

It is a vector space by definition in the usual way of addition and multiplication by scalars. Given $A \in \mathcal{S}_{\mathrm{d\text{-}i}}$, $A = \sum_{i=-m}^{n} a_i \int^{i}$, $p_A(s) := \sum_{i=-m}^{n} a_i s^i$ is the Laurent polynomial associated to A. On the other hand, given $a \in \mathcal{P}_{\mathrm{L}}$, $a = \sum_{i=m}^{n} a_i s^i$, the associated differential–integral operator is $a(\int) := \sum_{i=m}^{n} a_i \int^{i}$. Then the following identities hold: given $A \in \mathcal{S}_{\mathrm{d\text{-}i}}$ then $p_A(\int) = A$; given $a \in \mathcal{P}_{\mathrm{L}}$ then $p_{a(\int)} = a$. For differential–integral operators the following result can be established.

Theorem 2. *Let $A, B \in \mathcal{S}_{\mathrm{d\text{-}i}}$, p_A and p_B be the associated Laurent polynomials, and $\lambda \in \mathbb{R}(\mathbb{C})$. Then*

(a) *$A = B$ if and only if $p_A = p_B$,*
(b) *$p_{A+B} = p_A + p_B$,*
(c) *$p_{\lambda A} = \lambda \cdot p_A$.*

The above result is strictly analogous to Theorem 1 except for the absence of the statement (d). This is not surprising. In $\mathcal{S}_{\mathrm{d\text{-}i}}$ – differently from $\mathcal{S}_{\mathrm{d}}$ and $\mathcal{S}_{\mathrm{i}}$ – a composition of operators is not an operator of $\mathcal{S}_{\mathrm{d\text{-}i}}$. However, a weaker but useful algebraic result on operator composition in $\mathcal{S}_{\mathrm{d\text{-}i}}$ can be introduced.

Proposition 1. *Let $A = \sum_{i=-m_1}^{n_1} a_i \int^{i} \in \mathcal{S}_{\mathrm{d\text{-}i}}$, $B = \sum_{i=-m_2}^{n_2} b_i \int^{i} \in \mathcal{S}_{\mathrm{d\text{-}i}}$, and $f \in C_{\mathrm{p}}^{\infty} \cap C^{m_2-1}$. Then there exists $w \in \mathcal{P}$ with $\deg w \leq n_1 - 1$ such that*

$$A(Bf)(t) = (p_A \cdot p_B)\Big(\int\Big) f(t) - w(t),\ t \in \mathbb{R} \setminus S_f^{(m_1+m_2)}. \tag{5}$$

Proof of Proposition 1 is omitted for brevity. Proofs of other formal results presented in this section are reported in the Appendix.

3. The behavior of a system

Let us consider a linear, time-invariant system Σ with input $u \in C_{\mathrm{p}}^{\infty}(\mathbb{R})$ and output $y \in C_{\mathrm{p}}^{\infty}(\mathbb{R})$. The transfer function of Σ be given by

$$H(s) = \frac{b(s)}{a(s)} = \frac{b_m s^m + \cdots + b_1 s + b_0}{a_n s^n + \cdots + a_1 s + a_0}.$$

Polynomials $a(s)$ and $b(s)$ have real coefficients and are coprime (i.e. no common roots between them). Also assume $a_n \neq 0, b_m \neq 0$, and $m \leq n$. The order and relative degree of Σ are n and $r := n - m$ respectively. The differential equation associated to Σ is

$$\sum_{i=0}^{n} a_i D^i y(t) = \sum_{i=0}^{m} b_i D^i u(t) \tag{6}$$

or $a(D)\, y(t) = b(D)\, u(t)$ in compact notation. A *strong solution* of (6) is a pair $(u, y) \in C_{\mathrm{p}}^{\infty}(\mathbb{R})^2$ for which all the derivatives involved in (6) are well defined. As known, it is necessary to admit discontinuities in the input and the output, so that *weak solutions*, a more general concept of solution, are introduced.

Definition 5 (*Weak Solution*). A pair $(u, y) \in C_{\mathrm{p}}^{\infty}(\mathbb{R})^2$ is a weak solution of differential equation (6) if there exists a polynomial $g \in \mathcal{P}$ with $\deg g \leq n - 1$ such that the integral equation

$$\sum_{i=0}^{n} a_i \int^{n-i} y(t) = \sum_{i=0}^{m} b_i \int^{n-i} u(t) + g(t) \tag{7}$$

is satisfied for all $t \in \mathbb{R}$. Introduce the polynomials $A_n, B_n \in \mathcal{P}$ as $A_n(s) := s^n a(1/s)$, $B_n(s) := s^n b(1/s)$ and relation (7) can be compactly rewritten as

$$A_n(\textstyle\int) y(t) = B_n(\int) u(t) + g(t),\ t \in \mathbb{R}.$$

Remark 4. In (6) when $u \equiv 0$ or $y \equiv 0$ weak solutions of the resulting homogeneous equation $a(D)y(t) = 0$ or $b(D)u(t) = 0$ can be introduced in analogy with the above definition. For example $u \in C_{\mathrm{p}}^{\infty}(\mathbb{R})$ is a weak solution of $b(D)u(t) = 0$ if there exists $g \in \mathcal{P}$ with $\deg g \leq m - 1$ such that $\sum_{i=0}^{m} b_i \int^{m-i} u(t) = g(t),\, t \in \mathbb{R}$.

Evidently, in (6) a strong solution is also a weak solution but the converse does not hold. However, when restricted to homogeneous equations full equivalence holds.

Proposition 2. *A function $y \in C_p^\infty(\mathbb{R})$ is a strong solution of $a(D)y(t) = 0$ if and only if it is a weak solution.*

Proof. ($\Longrightarrow$) Apply the integral operator $\int^n$ to $a(D)y(t) = 0$ and by Lemma 4 and some algebraic manipulations we obtain:

$$A_n(\int)y(t) = \sum_{i=1}^{n}\sum_{j=0}^{i-1} \frac{a_i y^{(j)}(0)}{(j+n-i)!} t^{j+n-i}, \ \ t \in \mathbb{R},$$

where the polynomial on the right side has degree at most $n-1$. This proves that y is a weak solution of $a(D)y(t) = 0$.
($\Longleftarrow$) There exists $g \in \mathcal{P}$ with $\deg g \le n-1$ such that $A_n(\int)y(t) = g(t)$, $t \in \mathbb{R}$. Hence

$$y(t) = \frac{1}{a_n}\left(-\sum_{i=0}^{n-1}\int^{n-i} y(t) + g(t)\right), \ \ t \in \mathbb{R}. \tag{8}$$

By mathematical induction, it follows that $y \in C^l$, $l \in \mathbb{N}$, i.e. $y \in C^\infty$. First note that, by Lemma 1, $y \in C^0$ because all the addends of the right side of (8) are C^0-functions. Suppose now that $y \in C^l$ for any given $l \ge 0$. Then, by Lemma 1, it follows that all the addends of the right side of (8) are in C^{l+1}. Hence, $y \in C^{l+1}$ too. Now, apply the nth derivative D^n to $A_n(\int)y(t) = g(t)$, $t \in \mathbb{R}$. On the right side $D^n g(t) = 0$ and on the left side, by Lemma 2, $D^n A_n(\int)y(t) = a(D)y(t)$. Hence, $a(D)y(t) = 0$, $t \in \mathbb{R}$. □

Note that the introduced definition of weak solution is a simplified version of the general one reported in Polderman and Willems' book (Polderman & Willems, 1998). The behavior of Σ, i.e. the set of all pairs of input–output signals, can be then introduced as follows.

Definition 6 (*Behavior of Σ*). $\mathcal{B} := \{(u, y) \in C_p^\infty(\mathbb{R})^2 : (u, y)$ is a weak solution of (6)$\}$.

3.1. Behavior's properties

The linearity of Σ corresponds to $\mathcal{B}$ being a linear space by defining addition and multiplication by scalars in the obvious way. A first property on the continuity order of the system output is the following.

Proposition 3. *Let $(u, y) \in \mathcal{B}$, then $y \in C^{r-1}$.*

Proof. It is given by showing that $y \in C^q$, $q = -1, 0, \dots, r-1$ by mathematical induction. The case $q = -1$ is already proved because $y \in C_p^\infty(\mathbb{R}) = C^{r-1}$(cf. Definitions 2 and 6). When $r = 0$ we have already done, so consider $r > 0$. Hence suppose that $y \in C^q$ with $q < r-1$. From Definition 5 of weak solution, rewrite Eq. (7) as

$$a_n y(t) = -\sum_{i=0}^{n-1} a_i \int^{n-i} y(t) + \sum_{i=0}^{m} b_i \int^{n-i} u(t) + g(t). \tag{9}$$

In the right side of (9), the addend functions with the lower continuity orders are $a_{n-1}\int y$ and $b_m \int^r u$. By Lemma 1 $\int y \in C^{q+1}$, $\int^r u \in C^{r-1}$ and considering that $q+1 \le r-1$ we also have $\int^r u \in C^{q+1}$. Hence, by relation (9) $y \in C^{q+1}$ and this completes the inductive step. □

The following result gives necessary and sufficient conditions for an input–output pair to be a weak solution of (6).

Theorem 3. *A pair $(u, y) \in C_p^\infty(\mathbb{R})^2$ is a weak solution of (6), i.e. $(u, y) \in \mathcal{B}$, if and only if all the following conditions are satisfied:*

(1) $y \in C^{r-1}$;
(2) $S_y^{(r)} = S_u^{(0)}$;
(3) *there exists a polynomial $w \in \mathcal{P}$ with $\deg w \le m-1$ such that for all $t \in \mathbb{R} \setminus S_u^{(0)}$:*

$$\sum_{i=1}^{r} a_{m+i} D^i y(t) + \sum_{i=0}^{m} a_i \int^{m-i} y(t) = \sum_{i=0}^{m} b_i \int^{m-i} u(t) + w(t), \tag{10}$$

or in compact notation

$$A_m(\int)y(t) = B_m(\int)u(t) + w(t)$$

where $A_m(s) := s^m a(1/s) \in \mathcal{P}_L$ and $B_m(s) := s^m b(1/s) \in \mathcal{P}$.

Proof. ($\Longrightarrow$) Condition (1) is proved by Proposition 3. First consider $r = 0$ (i.e. $m = n$). The definition of weak solution implies that ($t \in \mathbb{R}$)

$$a_n y(t) + \sum_{i=0}^{n-1} a_i \int^{n-i} y(t) = b_n u(t) + \sum_{i=0}^{n-1} b_i \int^{n-i} u(t) + g(t) \tag{11}$$

with $g(t)$ being a polynomial with $\deg g \le m-1$. The zero-order discontinuity sets in the left-side and right-side of (11) must coincide. In this equation the only addends that can introduce discontinuities are $a_n y(t)$ and $b_n u(t)$, hence $S_y^{(0)} = S_u^{(0)}$ (condition (2)). Eq. (11) becomes Eq. (10) by setting $w(t) := g(t)$ and, a fortiori, it holds for all $t \in \mathbb{R} \setminus S_u^{(0)}$ (condition (3)).

Consider now $r \ge 1$ and take the rth derivative of both sides of Eq. (7). Applying Lemma 2 to the left side leads to

$$\sum_{i=0}^{n} a_i D^r \int^{n-i} y(t) = \sum_{i=1}^{r} a_{m+i} D^i y(t) + \sum_{i=0}^{m} a_i \int^{m-i} y(t) \tag{12}$$

where, taking into account that $y \in C^{r-1}$, all the addends are continuous functions defined over $\mathbb{R}$ with the exception of $a_n D^r y(t)$ which is defined over $\mathbb{R} \setminus S_y^{(r)}$. Hence, the overall summation function in (12) can only be defined over $\mathbb{R} \setminus S_y^{(r)}$. Similarly, the rth derivative of the right side of (7) can be expressed as

$$\sum_{i=0}^{m} b_i D^r \int^{n-i} u(t) + D^r g(t) = \sum_{i=0}^{m} b_i \int^{m-i} u(t) + D^r g(t). \tag{13}$$

As $D^r \int^r u(t)$ is defined just over $\mathbb{R} \setminus S_u^{(0)}$ and $D^r \int^r u(t) = u(t)$, $t \in \mathbb{R} \setminus S_u^{(0)}$ (cf. Lemma 2), expression in (13) is then defined over $\mathbb{R} \setminus S_u^{(0)}$. The domains over which (12) and (13) are defined must coincide so that $\mathbb{R} \setminus S_y^{(r)} = \mathbb{R} \setminus S_u^{(0)}$. Hence $S_y^{(r)} = S_u^{(0)}$ (condition (2)). Statement (3) is obtained by equating (12) with (13) and by setting $w(t) := D^r g(t)$.
($\Longleftarrow$) First consider $r = 0$ (i.e. $m = n$). Eq. (10) becomes

$$\sum_{i=0}^{n} a_i \int^{n-i} y(t) = \sum_{i=0}^{n} b_i \int^{n-i} u(t) + w(t) \tag{14}$$

with $t \in \mathbb{R} \setminus S_u^{(0)}$ and $\deg w \le n-1$. Set $g(t) := w(t)$. Eq. (14) coincides with the integral equation (7) and it also holds for all $t \in \mathbb{R}$, i.e. $(u, y) \in \mathcal{B}$. This is obvious when $S_u^{(0)} = \emptyset$ so we consider the case $S_u^{(0)} \neq \emptyset$. Take the right limit to $v \in S_u^{(0)} = S_y^{(0)}$ (cf. condition (2)) of both sides of (14):

$$a_n y(v^+) + \sum_{i=0}^{n-1} a_i \int^{n-i} y(v) = b_n u(v^+) + \sum_{i=0}^{n-1} b_i \int^{n-i} u(v) + g(v)$$

Taking into account that $y(v) = y(v^+)$, $u(v) = u(v^+)$ by convention in $C_p^\infty(\mathbb{R})$ (cf. Definition 2) it follows that Eq. (14) also holds when $t \in S_u^{(0)}$. Hence it holds for all $t \in \mathbb{R}$.

Now consider $r \geq 1$ and apply the integral operator $\int^r$ to Eq. (10) to obtain

$$\sum_{i=1}^{r} a_{m+i} \int^r D^i y(t) + \sum_{i=0}^{m} a_i \int^{n-i} y(t) = \sum_{i=0}^{m} b_i \int^{n-i} u(t) + \int^r w(t). \tag{15}$$

Eq. (15) holds for all $t \in \mathbb{R}$ because given any $f \in C_p^\infty$ (not necessarily defined all over $\mathbb{R}$) it follows that $\int f \in C_p^\infty(\mathbb{R})$ and evidently both sides of (10) are functions of C_p^∞. By condition (1) and Lemma 4 we have ($r \geq i$ and $y \in C^{i-1}$)

$$\int^r D^i y(t) = \int^{r-i} y(t) - \sum_{k=0}^{i-1} \frac{f^{(k)}(0)}{(k+r-i)!} t^{k+r-i}.$$

Hence, by change of a summation index and some algebraic manipulation equation (15) becomes

$$\sum_{j=m+1}^{n} a_j \int^{n-j} y(t) + \sum_{i=0}^{m} a_i \int^{n-i} y(t) = \sum_{i=0}^{m} b_i \int^{n-i} u(t) + \sum_{j=m+1}^{n} \sum_{k=0}^{j-m-1} \frac{a_j f^{(k)}(0)}{(k+n-j)!} t^{k+n-j} + \int^r w(t). \tag{16}$$

The polynomial defined by the double summation in (16) has degree at most $r-1$ and polynomial $\int^r w(t)$ has degree at most $n-1$ (cf. condition (3)). By $r \leq n$ the sum of these two polynomials which we can define as $g(t)$ has $\deg g \leq n-1$. This makes evident that (16) is the integral equation (7) for which $(u, y) \in \mathcal{B}$. □

A property of input–output pairs of the behavior $\mathcal{B}$ which is relevant for the inversion-based feedforward control is the following.

Proposition 4. *Consider a pair $(u, y) \in \mathcal{B}$. Let $p \in \mathbb{Z}$ with $p \geq -1$. Then $u \in C^p$ if and only if $y \in C^{r+p}$.*

The necessity part of Proposition 4 was first stated in Polderman and Willems (1998, p. 112) and subsequently in Piazzi and Visioli (2001c) the complete statement was presented without a formal proof.

Proof. ($\Longrightarrow$) Assume $u \in C^p$ and rewrite Eq. (7) as follows:

$$a_n y(t) = -\sum_{i=1}^{n} a_{n-i} \int^i y(t) + \sum_{i=r}^{n} b_{n-i} \int^i u(t) + g(t). \tag{17}$$

By Proposition 3 $y \in C^{r-1}$. Using mathematical induction it is then proved that $y \in C^q$ for $q = r-1, \ldots, r+p$. The base case ($q = r-1$) is already verified so that assume $y \in C^q$ with $q+1 \leq r+p$. Then we prove $y \in C^{q+1}$. In the right-hand side of (17) the functions with least continuity order are $a_{n-1} \int y$ and $b_m \int^r u$. Respectively these are a C^{q+1}-function and C^{r+p}-function by Lemma 1. Hence surely $a_n y \in C^{p+1}$ because $q+1 \leq r+p$ and this implies $y \in C^{q+1}$.
($\Longleftarrow$) Assume $y \in C^{r+p}$ and consider $p \geq 0$ because when $p = -1$, the sought implication , i.e. $u \in C^p$, is obviously verified. From $y \in C^{r+p}$ then $y \in C^r$, hence $S_y^{(r)} = \emptyset$ and from $S_u^{(0)} = S_y^{(r)}$ (cf. condition (2) of Theorem 3) $u \in C^0$. We then prove $u \in C^q$, $q = 0, 1, \ldots, p$ by using mathematical induction. The base case $q = 0$ is already proved so that suppose $u \in C^q$ with $q+1 \leq p$. Eq. (10) of Theorem 3 can be rewritten as

$$b_m u(t) = \sum_{i=1}^{r} a_{m+i} D^i y(t) + \sum_{i=0}^{m} a_i \int^{m-i} y(t) - \sum_{i=0}^{m-1} b_i \int^{m-i} u(t) - w(t) \tag{18}$$

and it holds for all $t \in \mathbb{R}$ because $S_y^{(r)} = \emptyset$. In the right-hand side of Eq. (18), the addend functions of least continuity order are $-b_{m-1} \int u$ and $a_n D^r y$ which are a C^{q+1}-function (from Lemma 1) and a C^p-function (because $y \in C^{r+p}$) respectively. By the fact that $C^p \subseteq C^{q+1}$ it follows that $b_m u(t) \in C^{q+1}$, i.e. $u \in C^{q+1}$. □

3.2. Behavior's representations

Poles and zeros of Σ play a major role in the following. Consider the factorizations (without loss of generality $a_n := 1$)

$$a(s) = \prod_{i=1}^{d} (s-p_i)^{\mu_i}, \quad b(s) = b_m \prod_{i=1}^{e} (s-z_i)^{\nu_i} \tag{19}$$

where $p_i \in \mathbb{C}$ with multiplicity μ_i, $i = 1, \ldots, d$ and $z_i \in \mathbb{C}$ with multiplicity ν_i, $i = 1, \ldots, e$ are the distinct poles and zeros of Σ respectively. The next definition introduces *pole and zero modes*, which are characteristic functions of Σ, defined over the entire time-axis ($\mathbb{R}$).

Definition 7 (*Pole and Zero Modes of Σ*)**.** Given a real (complex) pole $p \in \mathbb{R}$ ($p = \sigma \pm j\omega \in \mathbb{C}$) with multiplicity μ, the associated modes are $e^{pt}, te^{pt}, \ldots, t^{\mu-1}e^{pt}$ ($e^{\sigma t}\cos(\omega t), e^{\sigma t}\sin(\omega t), \ldots, t^{\mu-1}e^{\sigma t}\cos(\omega t), t^{\mu-1}e^{\sigma t}\sin(\omega t)$). All the pole modes of Σ are denoted by $m_i^P(t)$, $i = 1, \ldots, n$.

Given a real (complex) zero $z \in \mathbb{R}$ ($z = \rho \pm j\psi \in \mathbb{C}$) with multiplicity ν, the associated modes are $e^{zt}, te^{zt}, \ldots, t^{\nu-1}e^{zt}$ ($e^{\rho t}\cos(\psi t), e^{\rho t}\sin(\psi t), \ldots, t^{\nu-1}e^{\rho t}\cos(\psi t), t^{\nu-1}e^{\psi t}\sin(\psi t)$). All the zero modes of Σ are denoted by $m_i^Z(t)$, $i = 1, \ldots, m$.

Let us introduce the following lemmas (proofs are omitted for brevity).

Lemma 5. *The set of all weak solutions of $a(D)y(t) = 0$ and $b(D)u(t) = 0$ are respectively given by $\{y \in C^\infty : y(t) = \sum_{i=1}^{n} f_i m_i^P(t), f_i \in \mathbb{R}\}$ and $\{u \in C^\infty : u(t) = \sum_{i=1}^{m} g_i m_i^Z(t), g_i \in \mathbb{R}\}$.*

Lemma 6. *Let $f \in C_p^\infty(\mathbb{R})$, $k \in \mathbb{N}^+$ and $\lambda \in \mathbb{R}(\mathbb{C})$. Define*

$$g_k(t) := \int_0^t \frac{(t-v)^{k-1}}{(k-1)!} e^{\lambda(t-v)} f(v) dv, \ t \in \mathbb{R}.$$

Then

$$\left(1 - \lambda \int\right)^k g_k(t) = \int^k f(t), \ t \in \mathbb{R}. \tag{20}$$

For any $u \in C_p^\infty(\mathbb{R})$, the next result gives a particular (weak) solution of differential equation $a(D)y(t) = b(D)u(t)$. It is expressed by the convolution of u with $h(t)$, the analytical extension over $\mathbb{R}$ of $\mathcal{L}^{-1}[H(s)]$ (note that $h(t)1(t)$ is the classic impulse response of Σ).

Proposition 5 (*Output's Particular Solution*)**.** *Let $u \in C_p^\infty(\mathbb{R})$ and define*

$$y_{\text{par}}(t) := \int_0^t h(t-v)u(v)dv, \ t \in \mathbb{R}.$$

Then $A_n(\int)y_{\text{par}}(t) = B_n(\int)u(t)$, $t \in \mathbb{R}$ and $(u, y_{\text{par}}) \in \mathcal{B}$.

Proof. The partial fraction decomposition of $H(s)$ is

$$H(s) = \frac{b(s)}{a(s)} = \beta_0 + \sum_{i=1}^{d} \sum_{j=1}^{\mu_i} \frac{b_{ij}}{(s-p_i)^j} \tag{21}$$

and by inverse Laplace transform and analytic continuation ($\delta(t)$ is the delta Dirac function)

$$h(t) = \beta_0 \delta(t) + \sum_{i=1}^{d} \sum_{j=1}^{\mu_i} \frac{b_{ij}}{(j-1)!} t^{j-1} e^{p_i t}, \ t \in \mathbb{R}.$$

Hence, $y_{\mathrm{par}}(t) = \beta_0 u(t) + \sum_{i=1}^{d}\sum_{j=1}^{\mu_i} b_{ij} y_{ij}(t)$ where

$$y_{ij}(t) = \int_0^t \frac{(t-v)^{j-1}}{(j-1)!} e^{p_i(t-v)} u(v)dv,\ t \in \mathbb{R}.$$

The factorization of $a(s)$ in (19) and the partial fraction decomposition in (21) permit to express

$$b(s) = \beta_0 a(s) + \sum_{i=1}^{d}\sum_{j=1}^{\mu_i} b_{ij}[\prod_{k\neq i}^{d}(s-p_k)^{\mu_k}](s-p_i)^{\mu_i - j}.$$

From $B_n(s) = s^n b(1/s)$ and $A_n(s) = s^n a(1/s)$ (cf. Definition 5) and relation $n = \sum_{i=1}^{d}\mu_i$ it follows the identity

$$B_n(s) = \beta_0 A_n(s) + \sum_{i=1}^{d}\sum_{j=1}^{\mu_i} b_{ij}[\prod_{k\neq i}^{d}(1-p_k s)^{\mu_k}](1-p_i s)^{\mu_i - j} s^j. \tag{22}$$

Now compute $A_n(\int)y_{\mathrm{par}}(t)$ as follows

$$\begin{aligned} A_n(\int)y_{\mathrm{par}}(t) &= A_n(\int)[\beta_0 u(t) + \sum_{i=1}^{d}\sum_{j=1}^{\mu_i} b_{ij} y_{ij}(t)] \\ &= \beta_0 A_n(\int)u(t) \\ &+ \sum_{i=1}^{d}\sum_{j=1}^{\mu_i} b_{ij}[\prod_{k\neq i}^{d}(1-p_k\int)^{\mu_k}](1-p_i\int)^{\mu_i-j}(1-p_i\int)^j y_{ij}(t) \\ &= \beta_0 A_n(\int)u(t) \\ &+ \sum_{i=1}^{d}\sum_{j=1}^{\mu_i} b_{ij}[\prod_{k\neq i}^{d}(1-p_k\int)^{\mu_k}](1-p_i\int)^{\mu_i-j}\int^j u(t) \end{aligned}$$

where the last equality derives from $(1-p_i\int)^j y_{ij}(t) = \int^j u(t)$ as stated by Lemma 6. The algebraic identity (22) corresponds to an analogous integral operator identity. Hence, by the above expression of $A_n(\int)y_{\mathrm{par}}(t)$ it follows that $A_n(\int)y_{\mathrm{par}}(t) = B_n(\int)u(t)$, $t \in \mathbb{R}$. By Definition 5, we then have $(u, y_{\mathrm{par}}) \in \mathcal{B}$. □

The following theorem gives the input–output representation of the behavior $\mathcal{B}$.

Theorem 4 (*Input–output Representation*). *Define the following set*

$$\mathcal{B}_{\mathrm{i/o}} := \{(u,y) \in C_{\mathrm{p}}^{\infty}(\mathbb{R})^2 : y(t) = \int_0^t h(t-v)u(v)dv + \sum_{i=1}^{n} f_i m_i^P(t),\ t\in\mathbb{R},\ f_i \in \mathbb{R}\}.$$

Then $\mathcal{B}_{\mathrm{i/o}} = \mathcal{B}$.

Proof. ($\mathcal{B}_{\mathrm{i/o}} \subseteq \mathcal{B}$) Consider $(u, y) \in \mathcal{B}_{\mathrm{i/o}}$. By Proposition 5, $A_n(\int)\int_0^t h(t-v)u(v)dv = B_n(\int)u(t)$. On the other hand, by Lemma 5, $\sum_{i=1}^{n} f_i m_i^P(t)$ is a weak solution of $a(D)y(t) = 0$, so there exists $g \in \mathcal{P}$ with $\deg g \le n-1$ for which $A_n(\int)\sum_{i=1}^{n} f_i m_i^P(t) = g(t)$. Then,

$$\begin{aligned} A_n(\int)y(t) &= A_n(\int)\int_0^t h(t-v)u(v)dv \\ &+ A_n(\int)\sum_{i=1}^{n} f_i m_i^P(t) = B_n(\int)u(t) + g(t). \end{aligned}$$

Hence, $(u, y) \in \mathcal{B}$.

($\mathcal{B} \subseteq \mathcal{B}_{\mathrm{i/o}}$) Let $(u, y) \in \mathcal{B}$. Then there exists $g \in \mathcal{P}$ with $\deg g \le n-1$ such that $A_n(\int)y(t) = B_n(\int)u(t) + g(t)$. By Proposition 5, $A_n(\int)\int_0^t h(t-v)u(v)dv = B_n(\int)u(t)$. Hence, it follows that $A_n(\int)(y(t) - \int_0^t h(t-v)u(v)dv) = g(t)$. This means that $y(t) - \int_0^t h(t-v)u(v)dv$ is a weak solution of $a(D)y(t) = 0$. By Lemma 5, there exist coefficients $f_i \in \mathbb{R}$, $i = 1, \ldots, n$ for which $y(t) - \int_0^t h(t-v)u(v)dv = \sum_{i=1}^{n} f_i m_i^P(t)$. Thus, $(u, y) \in \mathcal{B}_{\mathrm{i/o}}$. □

Given any $y \in C_{\mathrm{p}}^{\infty} \cap C^{r-1}$, the next result (Proposition 6) gives a particular weak solution of $a(D)y(t) = b(D)u(t)$. From a system viewpoint, this entails to invert system Σ to obtain an input that causes the given output y. Hence, we introduce the inverse system of Σ by its transfer function $H^{-1}(s)$. By polynomial division $a(s) = q(s)b(s) + c(s)$ where $\deg c \le m-1$ and

$$q(s) = q_r s^r + q_{r-1}s^{r-1} + \cdots + q_0. \tag{23}$$

Therefore $H^{-1}(s) = a(s)/b(s) = q(s) + H_0(s)$ where $H_0(s) := c(s)/b(s)$ represents the so-called *zero dynamics* of Σ (Isidori, 1995). Also introduce $h_0(t)$, the analytical extension over $\mathbb{R}$ of $\mathcal{L}^{-1}[H_0(s)]$.

Proposition 6 (*Input's Particular Solution*). *Let $y \in C_{\mathrm{p}}^{\infty}(\mathbb{R}) \cap C^{r-1}$ and define*

$$u_{\mathrm{par}}(t) := q(D)y(t^+) + \int_0^t h_0(t-v)y(v)dv,\ t \in \mathbb{R}. \tag{24}$$

Then

$$A_m(\int)y(t) = B_m(\int)u_{\mathrm{par}}(t) + w(t),\ t \in \mathbb{R}\setminus S_y^{(r)} \tag{25}$$

where $w \in \mathcal{P}$ with $\deg w \le m-1$ is a suitable polynomial and $(u_{\mathrm{par}}, y) \in \mathcal{B}$.

Proof. This proof relies on Theorem 3. The partial fraction decomposition of $H_0(s)$ is

$$H_0(s) = \frac{c(s)}{b(s)} = \sum_{i=1}^{e}\sum_{j=1}^{\nu_i} \frac{c_{ij}}{(s-z_i)^j} \tag{26}$$

and by inverse Laplace transform and analytical continuation

$$h_0(t) = \sum_{i=1}^{e}\sum_{j=1}^{\nu_i} \frac{c_{ij}}{(j-1)!} t^{j-1} e^{z_i t},\ t \in \mathbb{R}.$$

From (24) it follows that $u_{\mathrm{par}}(t) = q(D)y(t^+) + \sum_{i=1}^{e}\sum_{j=1}^{\nu_i} c_{ij} u_{ij}(t)$ where

$$u_{ij}(t) := \int_0^t \frac{(t-v)^{j-1}}{(j-1)!} e^{z_i(t-v)} y(v)dv,\ t \in \mathbb{R}.$$

Factorize $B_m(s) = s^m b(1/s)$ as $B_m(s) = b_m \prod_{k=1}^{e}(1-z_k s)^{\nu_k}$ (cf. (19)) and, by Theorem 1, $B_m(\int) = b_m \prod_{k=1}^{e}(1-z_k\int)^{\nu_k}$. Then

$$\begin{aligned} B_m(\int)u_{\mathrm{par}}(t) &= B_m(\int)q(D)y(t^+) \\ &+ \sum_{i=1}^{e}\sum_{j=1}^{\nu_i} c_{ij} b_m \prod_{k=1}^{e}(1-z_k\int)^{\nu_k} u_{ij}(t) \\ &= B_m(\int)q(D)y(t^+) \\ &+ b_m \sum_{i=1}^{e}\sum_{j=1}^{\nu_i} c_{ij}[\prod_{k\neq i}^{e}(1-z_k\int)^{\nu_k}](1-z_i\int)^{\nu_i-j}(1-z_i\int)^j u_{ij}(t) \\ &= B_m(\int)q(D)y(t^+) \\ &+ b_m \sum_{i=1}^{e}\sum_{j=1}^{\nu_i} c_{ij}[\prod_{k\neq i}^{e}(1-z_k\int)^{\nu_k}](1-z_i\int)^{\nu_i-j}\int^j y(t),\ t \in \mathbb{R}. \end{aligned} \tag{27}$$

The last equality above is due to the identity $(1 - z_i \int)^j u_{ij}(t) = \int^j y(t)$ (cf. Lemma 6).

From $a(s) = q(s)b(s) + c(s)$ and $Q_0(s) := q(1/s) \in \mathcal{P}_\mathrm{L}$, $C_m(s) := s^m c(1/s) \in \mathcal{P}$ we have

$$A_m(s) = B_m(s)Q_0(s) + C_m(s).$$

This implies $A_m(\int) = (B_m \cdot Q_0)(\int) + C_m(\int)$ by virtue of Theorem 2. Hence,

$$A_m(\int)y(t) = (B_m \cdot Q_0)(\int)y(t) + C_m(\int)y(t), \ t \in \mathbb{R} \setminus S_y^{(r)}. \quad (28)$$

Considering that $y \in C_\mathrm{p}^\infty \cap C^{r-1}$, Proposition 1 can be recalled to express the composition of $Q_0(\int)$ with $B_m(\int)$ applied to $y(t)$. Hence, there exists $w \in \mathcal{P}$ with $\deg w \le m - 1$ such that

$$B_m(\int)(Q_0(\int)y(t)) = (B_m \cdot Q_0)(\int)y(t) - w(t), \quad (29)$$

$t \in \mathbb{R} \setminus S_y^{(r)}$. From (28) and (29) and taking into account that $Q_0(\int) = q(D)$ it follows that

$$A_m(\int)y(t) = B_m(\int)(q(D)y(t)) + C_m(\int)y(t) + w(t), \quad (30)$$

$t \in \mathbb{R} \setminus S_y^{(r)}$. From (26) and (19) $c(s) = b_m \sum_{i=1}^{e} \sum_{j=1}^{\nu_i} c_{ij} \, [\prod_{k\neq i}^{e}(s - z_k)^{\nu_k}](s - z_i)^{\nu_i - j}$ and $C_m(s) = b_m \sum_{i=1}^{e} \sum_{j=1}^{\nu_i} c_{ij}[\prod_{k\neq i}^{e}(1 - z_k s)^{\nu_k}](1 - z_i s)^{\nu_i - j} s^j$ so that, by Theorem 1, operator $C_m(\int)$ can be expressed as

$$C_m(\int) = b_m \sum_{i=1}^{e} \sum_{j=1}^{\nu_i} c_{ij}[\prod_{k\neq i}^{e}(1 - z_k \int)^{\nu_k}](1 - z_i \int)^{\nu_i - j} \int^j.$$

Therefore, by comparing (27) with (30) it follows the differential–integral relation (25). Considering that $y \in C_\mathrm{p}^\infty \cap C^{r-1}$ and

$$q(D)y(t^+) = q_r D^r y(t^+) + q_{r-1} D^{r-1} y(t) + \cdots + q_0 y(t),$$

$t \in \mathbb{R}$, by definition (24) it follows that $S_y^{(r)} = S_{u_\mathrm{par}}^{(0)}$. Hence, all the conditions of Theorem 3 are satisfied and the proof is thus complete. □

The output–input (or inverse) representation of the behavior $\mathcal{B}$ can be then introduced as follows.

Theorem 5 (*Output–input Representation*). *Define the following set*

$$\mathcal{B}_{\mathtt{o/i}} := \{(u, y) \in C_\mathrm{p}^\infty(\mathbb{R})^2 : y \in C^{r-1}, u(t) = q(D)y(t^+) + \int_0^t h_0(t - v)y(v)dv + \sum_{i=1}^{m} g_i m_i^Z(t), \ t \in \mathbb{R}, \ g_i \in \mathbb{R}\}.$$

Then $\mathcal{B}_{\mathtt{o/i}} = \mathcal{B}$.

Proof. ($\mathcal{B}_{\mathtt{o/i}} \subseteq \mathcal{B}$) Consider $(u, y) \in \mathcal{B}_{\mathtt{o/i}}$. By Proposition 6 there exists $w \in \mathcal{P}$ with $\deg w \le m - 1$ such that

$$A_m(\int)y(t) = B_m(\int)[q(D)y(t^+) + \int_0^t h_0(t - v)y(v)dv] + w(t), t \in \mathbb{R} \setminus S_y^{(r)}. \quad (31)$$

On the other hand, by Lemma 5, $\sum_{i=1}^{m} g_i m_i^Z(t)$ is a weak solution of $b(D)u(t) = 0$, so there exists $g \in \mathcal{P}$ with $\deg g \le m - 1$ such that

$$B_m(\int) \sum_{i=1}^{m} g_i m_i^Z(t) = g(t), \ t \in \mathbb{R}. \quad (32)$$

Then set $x := w + g$ and thus $\deg x \le m - 1$ and by the sum of (31) and (32) obtain

$$A_m(\int)y(t) = B_m(\int)u(t) + x(t), \ t \in \mathbb{R} \setminus S_y^{(r)}.$$

From $y \in C_\mathrm{p}^\infty \cap C^{r-1}$ and the definition of $u(t)$ it follows that $S_y^{(r)} = S_u^{(0)}$. Hence, by Theorem 3 $(u, y) \in \mathcal{B}$.

($\mathcal{B} \subseteq \mathcal{B}_{\mathtt{o/i}}$) Let $(u, y) \in \mathcal{B}$. Then, by Theorem 3, $y \in C^{r-1}$, $S_y^{(r)} = S_u^{(0)}$ and there exists $w_1 \in \mathcal{P}$ with $\deg w_1 \le m - 1$ such that

$$A_m(\int)y(t) = B_m(\int)u(t) + w_1(t), \ t \in \mathbb{R} \setminus S_y^{(r)}. \quad (33)$$

By Proposition 6

$$A_m(\int)y(t) = B_m(\int)(q(D)y(t^+) + \int_0^t h_0(t - v)y(v)dv) + w_2(t), \ t \in \mathbb{R} \setminus S_y^{(r)} \quad (34)$$

with $w \in \mathcal{P}$ and $\deg w_2 \le m - 1$. From (33) and (34) we obtain

$$B_m(\int)[u(t) - q(D)y(t^+) - \int_0^t h_0(t - v)y(v)dv] = g(t), \quad (35)$$

$t \in \mathbb{R} \setminus S_y^{(r)}$ with $g := w_2 - w_1$ and $\deg g \le m - 1$. Both sides of relation (35) are functions of $C_\mathrm{p}^\infty(\mathbb{R})$ so that relation (35) holds for all $t \in \mathbb{R}$. Hence $u(t) - q(D)y(t^+) - \int_0^t h_0(t - v)y(v)dv$ is a weak solution of $d(D)u(t) = 0$ so that, by Lemma 5,

$$u(t) - q(D)y(t^+) - \int_0^t h_0(t - v)y(v)dv = \sum_{i=1}^{m} g_i m_i^Z(t), \ t \in \mathbb{R}$$

with $g_i \in \mathbb{R}$. So, it follows $(u, y) \in \mathcal{B}_{\mathtt{o/i}}$. □

Remark 5. Differently from the classic behavior theory (Polderman & Willems, 1998), the theory herein presented does not require topological notions to be introduced such as e.g. those of measure and convergence. This significantly expedites the deduction of results. Indeed, all the provided proofs are direct without the need, e.g., to find a property first for strong solutions and then, by using convergence arguments, for weak solutions.

4. Inversion-based control

The following definitions regards the characterization of a signal.

Definition 8 (*Polynomial Order*). A signal $f \in C_\mathrm{p}^\infty$ has polynomial order $l \in \mathbb{N}$ if there exist constants $M > 0$ and $N > 0$ such that

$$|f(t^+)| \le M\,|t|^l + N, \forall t \in \mathbb{R}.$$

Definition 9 (*Smoothness Degree*). A signal $f \in C_\mathrm{p}^\infty(\mathbb{R})$ is said to have smoothness degree $k \ge -1$ if $f \in C^k$ and $f \notin C^{k+1}$. Signal f has *infinite* smoothness, i.e. $k = \infty$, when $f \in C^\infty$.

Remark 6. In words, smoothness degree k of $f \in C_\mathrm{p}^\infty(\mathbb{R})$ means that f has continuity order k but does not have continuity order $k+1$ or, more specifically, $f^{(k+1)}$ has jump discontinuities (at least one over $\mathbb{R}$).

The following result – which is a direct consequence of Proposition 4 – relates the smoothness degrees of the input and the output of Σ.

Corollary 1. *Consider a pair* $(u, y) \in \mathcal{B}$. *Then, input* u *has smoothness degree* k *if and only if output* y *has smoothness degree* $k + r$.

Proof. ($\Longrightarrow$) Input u has smoothness (degree) k, i.e. $u \in C^k$ and $u \notin C^{k+1}$. By Proposition 4 then $y \in C^{k+r}$. On the other hand $y \notin C^{k+r+1}$. Indeed suppose, by absurd, that $y \in C^{k+r+1}$. Then, $u \in C^{k+1}$ by Proposition 4 and this contradicts the hypothesis.
($\Longleftarrow$) Output y has smoothness $k + r$, i.e. $y \in C^{k+r}$ and $y \notin C^{k+r+1}$. By Proposition 4 then $u \in C^k$. On the other hand $u \notin C^{k+1}$. Indeed suppose, by absurd, that $u \in C^{k+1}$. Then, $y \in C^{k+r+1}$ by Proposition 4 and this contradicts the hypothesis. □

The stable (input–output) inversion problem can be then introduced as follows.

Problem 1 (*Stable Inversion*)**.** Let be given a desired output signal $y_d \in C_{\mathrm{p}}^{\infty}(\mathbb{R})$ with smoothness degree k. Assume that y_d and its derivatives $y_d^{(1)}, \ldots, y_d^{(r)}$ have all polynomial order l. Find an (inverse) input $u_d \in C_{\mathrm{p}}^{\infty}(\mathbb{R})$ with polynomial order l such that $(u_d, y_d) \in \mathcal{B}$.

A straightforward necessary condition for Problem 1 to have a solution is $k \geq r-1$ (cf. (38) in Theorem 6). For minimum-phase systems, when this condition is satisfied by a causal desired output y_d we could prove that the input's particular solution (24) of Proposition 6 is actually a solution of Problem 1. This solution corresponds to perform on the causal y_d a standard input–output inversion. Note that, in this case, the corresponding inverse u_d is causal too. However, for nonminimum-phase systems, solution (24) breaks down because the convolution integral $\int_0^t h_0(t-v)y_d(v)dv$ diverges exponentially. A way to overcome this obstruction is to search for a noncausal (inverse) input. This was shown – in the case of y_d belonging to $L_1 \cap L_\infty$ or being a polynomial – by the pioneering works on inversion-based control (Bayo, 1987; Chen, 1993; Devasia et al., 1996; Hunt et al., 1996). They made the following necessary assumption.

Assumption 1. The zero dynamics of Σ is hyperbolic, i.e. there are no zeros of Σ on the imaginary axis of $\mathbb{C}$.

Taking into account this assumption, the solution to Problem 1, provided by the next result (Theorem 6), can be obtained by splitting the zero dynamics into stable and unstable parts. Factorize the zero polynomial of Σ as $b(s) = b_m b^-(s)b^+(s)$ where $b^-(s)$ and $b^+(s)$ are monic polynomials whose roots have all negative and positive real parts respectively. By partial fraction decomposition $H_0 = c(s)/b(s)$ can be split as $H_0(s) = H_0^-(s) + H_0^+(s)$ with $H_0^-(s) = c^-(s)/b^-(s)$, $H_0^+(s) = c^+(s)/b^+(s)$ being c^-, c^+ suitable polynomials. Then, $h_0^-(t)$ and $h_0^+(t)$ be respectively the analytical extension over $\mathbb{R}$ of $\mathcal{L}^{-1}[H_0^-(s)]$ and $\mathcal{L}^{-1}[H_0^+(s)]$ for which

$$h_0(t) = h_0^-(t) + h_0^+(t),\ t \in \mathbb{R}. \tag{36}$$

Denote by $m_i^-(t)$, $i = 1, \ldots, m^-$ and $m_i^+(t)$, $i = 1, \ldots, m^+$ the stable and unstable zero modes respectively ($m^- + m^+ = m$). Hence, there exist real coefficients α_i and β_i such that

$$h_0^-(t) = \sum_{i=1}^{m^-} \alpha_i m_i^-(t),\ \ h_0^+(t) = \sum_{i=1}^{m^+} \beta_i m_i^+(t),\ t \in \mathbb{R}. \tag{37}$$

Theorem 6. *The stable inversion problem (Problem 1) has a solution if and only if*

$$k \geq r-1, \tag{38}$$

(i.e. the smoothness degree of y_d is greater than or equal to the relative degree of Σ minus one). When condition (38) is satisfied the solution is unique and can be expressed as

$$u_d(t) = q(D)y_d(t^+) + \int_{-\infty}^{t} h_0^-(t-v)y_d(v)dv - \int_{t}^{+\infty} h_0^+(t-v)y_d(v)dv,\ t \in \mathbb{R}. \tag{39}$$

Proof. ($\Longrightarrow$) Suppose that $u_d \in C_{\mathrm{p}}^{\infty}(\mathbb{R})$ is a solution of Problem 1. Hence $(u_d, y_d) \in \mathcal{B}$ and by Proposition 3 $y_d \in C^{r-1}$. This implies that k, the smoothness degree of y_d must satisfy condition (38) (cf. Definition 9).

($\Longleftarrow$) Signal u_d can be rewritten as (cf. (23) and (37))

$$u_d(t) = \sum_{i=0}^{r} q_i D^i y_d(t^+) + \sum_{i=1}^{m^-} \alpha_i \int_{-\infty}^{t} m_i^-(t-v)y_d(v)dv - \sum_{i=1}^{m^+} \beta_i \int_{t}^{+\infty} m_i^+(t-v)y_d(v)dv. \tag{40}$$

All the addends in (40) have polynomial order l and so $u_d(t)$ too. Indeed, by the assumptions of Problem 1 functions $D^i y_d(t)$, $i = 0, 1, \ldots, r$ have all polynomial order l as well as the integrals $\int_{-\infty}^{t} m_i^-(t-v)y_d(v)dv$ and $\int_{t}^{+\infty} m_i^+(t-v)y_d(v)dv$. For example, consider the former integral and without loss of generality let $m_i^-(t) = t^p e^{\rho t}\cos(\psi t)$, $p \in \mathbb{N}$, $\rho < 0$, $\psi \in \mathbb{R}$:

$$\begin{aligned} &\left|\int_{-\infty}^{t} m_i^-(t-v)y_d(v)dv\right| \leq \int_{-\infty}^{t} (t-v)^p e^{\rho(t-v)}(M|v|^l + N)dv \\ &= \int_{0}^{+\infty} \xi^p e^{\rho\xi}(M|t-\xi|^l + N)d\xi \\ &\leq M\sum_{j=0}^{l}\binom{l}{j}|t|^{l-j}\frac{(p+j)!}{|\rho|^{p+j+1}} + N\frac{p!}{|\rho|^{p+1}}. \end{aligned} \tag{41}$$

From the last inequality in (41) it follows that $\int_{-\infty}^{t} m_i^-(t-v)y_d(v)dv$ has polynomial order l. A similar argument can prove the same for $\int_{t}^{+\infty} m_i^+(t-v)y_d(v)dv$.

Output $y_d \in C^k$ and by condition (38), $y_d \in C^{r-1}$. We will prove that $(u_d, y_d) \in \mathcal{B}_{\mathrm{o/i}}$, i.e. $(u_d, y_d) \in \mathcal{B}$ by Theorem 5. Taking into account relation (36) we have $\int_{-\infty}^{t} h_0^-(t-v)y_d(v)dv - \int_{t}^{+\infty} h_0^+(t-v)y_d(v)dv = \int_0^t h_0(t-v)y_d(v)dv + \int_{-\infty}^{0} h_0^-(t-v)y_d(v)dv - \int_{0}^{+\infty} h_0^+(t-v)y_d(v)dv$. Integrals $\int_{-\infty}^{0} h_0^-(t-v)y_d(v)dv$ and $\int_{0}^{+\infty} h_0^+(t-v)y_d(v)dv$ are linear combinations of stable and unstable zero modes respectively. Indeed, by (37) they can be expressed as $\sum_{i=1}^{m^-}\alpha_i \int_{-\infty}^{0} m_i^-(t-v)y_d(v)dv$ and $\sum_{i=1}^{m^+}\beta_i \int_{0}^{+\infty} m_i^+(t-v)y_d(v)dv$. Without loss of generality suppose that $m_i^-(t) = t^p e^{\rho t}\cos(\psi t)$, $p \in \mathbb{N}$, $\rho < 0$, $\psi \in \mathbb{R}$. Hence, $\int_{-\infty}^{0} m_i^-(t-v)y_d(v)dv = \sum_{j=0}^{p}\binom{p}{j} t^{p-j}e^{\rho t}\cos(\psi t)\int_{-\infty}^{0} e^{-\rho v}(-v)^j\cos(\psi v)y_d(v)dv + \sum_{j=0}^{p}\binom{p}{j} t^{p-j}e^{\rho t}\sin(\psi t)\int_{-\infty}^{0} e^{-\rho v}(-v)^j\sin(\psi v)y_d(v)dv$. The above improper integrals have finite values because $y_d(t)$ has finite polynomial order. Thus $\int_{-\infty}^{0} m_i^-(t-v)y_d(v)dv$ is a linear combination of the modes $e^{\rho t}\cos(\psi t), e^{\rho t}\sin(\psi t), \ldots, t^p e^{\rho t}\cos(\psi t), t^p e^{\rho t}\sin(\psi t)$. A similar argument runs for $\int_{0}^{+\infty} m_i^+(t-v)y_d(v)dv$. Eventually, we conclude that $u_d(t) = q(D)y_d(t^+) + \int_0^t h_0(t-v)y_d(v)dv + \sum_{i=1}^{m} g_i m_i^Z(t)$, $t \in \mathbb{R}$ where the g_i's are suitable real constants. By virtue of Theorem 5 it is then proved that $(u_d, y_d) \in \mathcal{B}$.

Suppose that there exists another solution $u_1(t)$ to Problem 1: $u_1(t)$ has polynomial order l and $(u_1, y_d) \in \mathcal{B}$. By Theorem 5 there exist real coefficients h_i, $i = 1, \ldots, m$ for which $u_1(t) = q(D)y_d(t^+) + \int_0^t h_0(t-v)y_d(v)dv + \sum_{i=1}^{m} h_i m_i^Z(t)$, $t \in \mathbb{R}$. The difference $u_d(t) - u_1(t) = \sum_{i=1}^{m}(g_i - h_i)m_i^Z(t)$ is then still a function with polynomial order l. By absurd, suppose that $u_1(t) \not\equiv u_d(t)$. Hence, there exists at least one index k such that $g_k \neq h_k$. In this case, $u_d(t) - u_1(t)$ cannot have polynomial order because it diverges exponentially. This contradiction concludes the proof. □

It may be interesting to note that behavior theory is not necessary in understanding and applying the solution proposed by Theorem 6. Nevertheless, the simplified behavior theory herein proposed appears indispensable for the rigorous and thorough deduction of this solution. Indeed, the inverse representation of the behavior $\mathcal{B}$ (cf. Theorem 5) is the major result leading directly to Theorem 6. Informally, we could add that behavior theory is a natural choice to address inversion-based control because in this theory – differently from state-space approaches – a generic input or output is a noncausal signal.

Theorem 6 clarifies two important features of inversion-based control: there exists a simple, necessary and sufficient condition for the stable inversion problem to have a solution (cf. (38)) and when this condition is satisfied the solution is unique. Both features were not addressed in previous works on inversion-based control (cf. Devasia et al., 1996; Hunt et al., 1996). The unique solution provided by the closed-form expression (39) could be also determined by the state-space inversion procedure presented in Hunt et al. (1996) (or Devasia et al. (1996), the procedures in Hunt et al. (1996) and Devasia et al. (1996) basically coincide when that in Devasia et al. (1996) is restricted to linear systems). Applying formula (39) requires to compute the transfer function inverse and the partial fraction decomposition of the zero transfer function whereas the procedure in Hunt et al. (1996) requires to compute the system normal form (cf. Isidori, 1995) and the Carathéodory solution (or Green function, cf. Devasia et al., 1996) of the zero dynamics. From a computational viewpoint, a possible advantage of (39) over the state-space procedure is the avoidance of eigenspace computations that are necessary in determining the Carathéodory solution.

Remark 7. It is worth noting that the standard inversion fails even for minimum-phase systems when the desired output is noncausal. Indeed, standard inversion is given by $u_d(t) = q(D)y_d(t^+) + \int_0^t h_0(t-v)y_d(v)dv$ (cf. (24)) that, in general, is unbounded over $\mathbb{R}$. This does not happen with stable inversion that, in this case, takes the simplified expression $u_d(t) = q(D)y_d(t^+) + \int_{-\infty}^t h_0(t-v)y_d(v)dv$ (cf. (39)).

Remark 8. Usually, stable inversion is applied to a bounded output and the emphasis is to obtain a bounded input (Devasia et al., 1996; Pallastrelli & Piazzi, 2005; Sogo, 2010; Zoua & Devasia, 1999). Theorem 6 shows that stable inversion can be extended to signals with finite polynomial order, i.e. polynomially unbounded signals (cf. Definition 8). This may be useful in certain applications such as, e.g., those in which ramp tracking is required (Peng, Singh, & Milano, 2015) (cf. Example 1 in Section 5).

Remark 9. The inversion formula (39) works for any causal or noncausal and polynomially unbounded y_d provided that the relative degree condition (38) be satisfied. In such a way, it generalizes the analogous formula proposed in Pallastrelli and Piazzi (2005). That required a bounded causal output y_d with a more stringent (greater) smoothness.

In applying the inversion-based control the relative degree condition (38) plays a crucial role. Indeed, in some cases, this condition can impede the direct application of the method because the desired output lacks a sufficiently high smoothness. A way to overcome this obstruction is to modify the desired output with a suitable smoothing (Costalunga & Piazzi, 2015).

Often, when dealing with control applications, the desired output is a causal signal. For nonminimum-phase systems, this implies that the inverse input exhibits the so-called *preaction* (or *preactuation*) *control* (Devasia et al., 1996; Marro & Piazzi, 1996).

Proposition 7 (*Preaction Control*). *The desired output $y_d \in C_p^\infty(\mathbb{R})$ be a causal signal ($y_d(t) = 0, t < 0$) and suppose that the relative degree condition (38) be satisfied. Then, the inverse input u_d, solution of the stable inverse problem for negative times is given by a linear combination of the unstable zero modes, i.e. there exist real coefficients γ_i such that*

$$u_d(t) = \sum_{i=1}^{m^+} \gamma_i m_i^+(t), \; t < 0. \tag{42}$$

Proof. From (39) and (37) we have $u_d(t) = -\int_0^{+\infty} h_0^+(t-v)y_d(v)dv = -\sum_{i=1}^{m^+}\beta_i \int_0^{+\infty} m_i^+(t-v)y_d(v)dv,\ t < 0$. With an argument similar to that used in the proof of Theorem 6, it can be shown that $\int_0^{+\infty} m_i^+(t-v)y_d(v)dv$ is a linear combination of a subset of the unstable zero modes of Σ. Hence, relation (42) follows. □

The time range of preaction control is $(-\infty, 0)$, but $u_d(t)$ decays exponentially to zero as $t \to -\infty$. This has suggested to truncate $u_d(t)$, i.e. to force $u_d(t) = 0$ whenever $t < t_p < 0$ where $|t_p|$ is the preaction time leading to an approximate inverse input (Perez & Devasia, 2003).

In many cases, in designing the output $y_d(t)$ we can distinguish a steady-state component that is reached when $t \geq \tau$ and a previous transient component with $t < \tau$ (being τ a given time instant). This leads to the phenomenon of *postaction* (or *postactuation*) *control* (Braschi, Marconi, & Melchiorri, 1998; Farid & Lukasiewicz, 1996) herein presented with an extended formulation.

Proposition 8 (*Postaction Control*). *Let be given a steady-state pair $(u_{ss}, y_{ss}) \in \mathcal{B}$ with u_{ss} having finite polynomial order and a desired output $y_d \in C_p^\infty(\mathbb{R})$ for which $y_d(t) = y_{ss}(t), t \geq \tau$ for a given $\tau \in \mathbb{R}$. Suppose that the relative degree condition (38) be satisfied. Then, the inverse input u_d, solution of the stable inversion problem, for $t \geq \tau$ is given by the steady-state input u_{ss} plus a linear combination of the stable zero modes, i.e. there exist real coefficients δ_i such that*

$$u_d(t) = u_{ss}(t) + \sum_{i=1}^{m^-} \delta_i m_i^-(t), \; t \geq \tau. \tag{43}$$

Proof. From $(u_{ss}, y_{ss}) \in \mathcal{B}$ and $(u_d, y_d) \in \mathcal{B}$ it follows that $(u_d - u_{ss}, y_d - y_{ss}) \in \mathcal{B}$. By Theorem 3, there exists $w \in \mathcal{P}$ with $\deg w \leq m-1$ satisfying for $t \in \mathbb{R} \setminus S_{y_d - y_{ss}}^{(r)}$

$$A_m(\int)(y_d(t) - y_{ss}(t)) = B_m(\int)(u_d(t) - u_{ss}(t)) + w(t). \tag{44}$$

Taking into account that $y_d(t) - y_{ss}(t) = 0, t \geq \tau$ it follows that equality (44) holds for any $t > \tau$ and $A_m(\int)(y_d(t) - y_{ss}(t))$ with $t > \tau$ is a polynomial whose degree is less or equal to $m-1$. Hence $u_d(t) - u_{ss}(t)$ is a weak solution of the homogeneous differential equation $b(D)u(t) = 0, t > \tau$ (cf. Remark 4). This implies that $u_d - u_{ss}$ is a linear combination of the zero modes of Σ (cf. Lemma 5). Note that $\{m_i^Z(t), i = 1, \ldots, m\} = \{m_i^-(t), i = 1, \ldots, m^-\} \cup \{m_i^+(t), i = 1, \ldots, m^+\}$ so that there exist $\delta_i \in \mathbb{R}$ and $\gamma_i \in \mathbb{R}$ such that

$$u_d(t) - u_{ss}(t) = \sum_{i=1}^{m^-} \delta_i m_i^-(t) + \sum_{i=1}^{m^+} \gamma_i m_i^+(t), \; t > \tau. \tag{45}$$

Since both u_d and u_{ss} have finite polynomial order, so is $u_d - u_{ss}$. Hence, all the γ_i's must be zero otherwise $u_d - u_{ss}$ would diverge exponentially as $t \to +\infty$ in contradiction with its finite polynomial order. Relation (45) also holds for $t = \tau$ obtaining (43) because $u_d - u_{ss}$ is right continuous (cf. Remark 1). □

5. Examples

Example 1. Consider the feedforward control of a flexible link rotating in a horizontal plane (see Fig. 1). One end of the link is fastened to a servo motor whose hub angle u is the system input

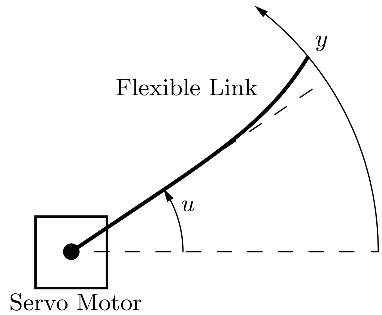

Fig. 1. Example 1: Sketch of the planar view of a rotary flexible link.

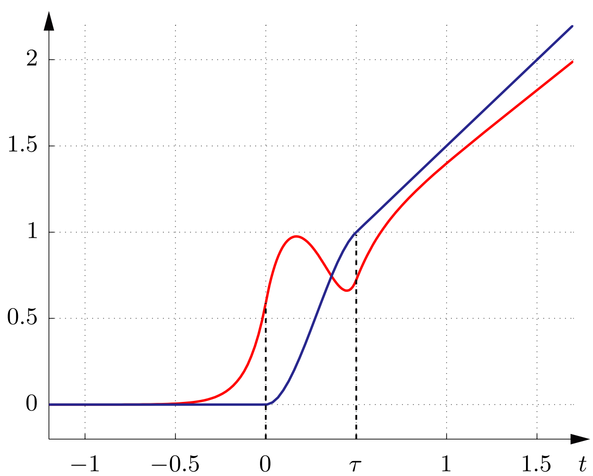

Fig. 2. Example 1: the output $\tilde{y}_d$ [m] with smoothness degree 2 (in blue) and the inverse input $\tilde{u}_d$ [rad] (in red). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

(measured in radians [rad]). The other end of the link is free and its position measured in meters [m] along the arc path is y, the system output. By linearizing the system dynamics (Geniele, Patel, & Khorasani, 1997) and by modeling the dominant flexible mode a nonminimum-phase second-order system can be considered. With data taken from Piazzi and Visioli (2001a), the system transfer function is $H_1(s) = -0.1913\frac{(s-9.31)(s+6.93)}{[(s+1.16)^2+2.99^2]}$ and its relative degree is $r = 0$.

The desired output be a step plus ramp function $y_d(t) = (1 + t)\,1(t)$, i.e. the link end-point is required to change its position and then to follow a ramp signal with velocity of 1 m/s. Both y_d and $y_d^{(1)}$ have polynomial order $l = 1$ (cf. Definition 8)) and the smoothness degree of y_d is $k = -1$ (cf. Definition 9). Hence, the stable inversion problem (cf. Problem 1) has solution because condition (38) $k \geq r - 1$ is satisfied. However, by Corollary 1, the corresponding inverse input would have smoothness degree -1. This would be unacceptable because a discontinuous hub angle cannot be physically implemented. For this application we have to require that the inverse input have smoothness degree 2 or greater, i.e. the hub angle position, velocity, and acceleration are all continuous functions. Therefore, we can resort to a polynomial smoothing on y_d (Costalunga & Piazzi, 2015) to obtain a smoothness degree 2 on a smoothed $\tilde{y}_d(t)$ defined as

$$\tilde{y}_d(t) = \begin{cases} 0 & t < 0 \\ p(t) & t \in [0, \tau] \\ y_d(t-\tau) & t > \tau. \end{cases} \tag{46}$$

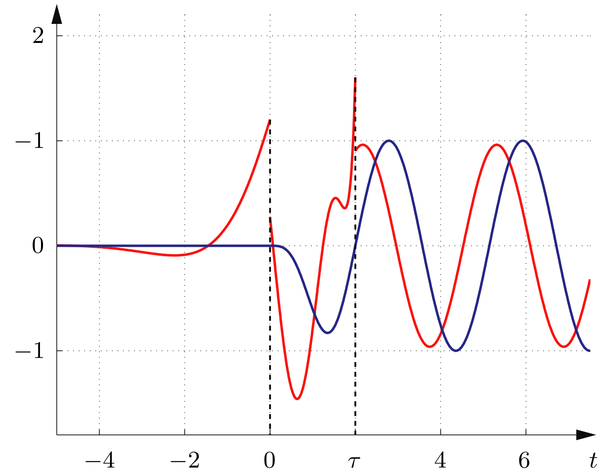

Fig. 3. Example 2: the output $\tilde{y}_d$ with smoothness degree 3 (in blue) and the inverse input $\tilde{u}_d$ (in red). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

The parameter τ is the smoothing time and $p(t)$ is an interpolating polynomial designed to obtain for $\tilde{y}_d$ the required smoothness degree. Also note that the polynomial orders of $\tilde{y}_d$ and its derivatives are the same of y_d and its derivatives for any interpolating $p(t)$. In this case, the polynomial $p(t)$ can be written as $p(t) = \sum_{i=0}^{2} y_d^{(i)}(0^+)\tau^i q_{2i}(t/\tau)$ where $q_{20}(v) = 10v^3 - 15v^4 + 6v^5$, $q_{21}(v) = -4v^3 + 7v^4 - 3v^5$, $q_{22}(v) = \frac{1}{2}v^3 - v^4 + \frac{1}{2}v^5$.

For example choose $\tau = 0.5$ s and apply the inversion formula (39) to obtain: $\tilde{u}_d(t) = 0.506066e^{9.31t}$, $t < 0$ (preaction control, cf. Proposition 7); $\tilde{u}_d(t) = -7.586147 + 61.566791t - 171.45818t^2 + 191.45423t^3 - 40.134658t^4 + 120.004756t^5 + 8.1400136e^{-6.93t} - 0.04779998e^{9.31t}$, $t \in [0, \tau]$; $\tilde{u}_d(t) = 0.573912 + 0.83336636t - 8.1232057e^{-6.93t}$, $t > \tau$. Note that, when $t > \tau$, the input can be written as $\tilde{u}_d(t) = u_{ss}(t) - 8.1232057e^{-6.93t}$ where $u_{ss}(t) = 0.573912 + 0.83336636t$ is the steady-state input (with $y_{ss}(t) = 0.5 + t$ being the corresponding steady-state output) and $-8.1232057e^{-6.93t}$ is the decaying postaction control (cf. Proposition 8). The pair $(\tilde{u}_d, \tilde{y}_d)$ is plotted in Fig. 2.

Example 2. Consider a system with relative degree $r = 4$ and transfer function $H_2(s) = 80\frac{(s+1)[(s-1)^2+1]}{(s+2)^5\left(s+\frac{1}{2}\right)^2}$ (cf. Costalunga & Piazzi, 2015). The desired output is $y_d(t) = \sin(2t)1(t)$ which has degree 0 of smoothness. It has polynomial order 0 and so have its derivatives (cf. Problem 1). Condition (38) is not satisfied, so the desired output is modified according to (46). An inverse input can be found by choosing a smoothness degree 3 for $\tilde{y}_d$. The polynomial interpolation scheme proposed in Costalunga and Piazzi (2015) leads to $p(t) = \sum_{i=0}^{3} y_d^{(i)}(0^+)\tau^i q_{3i}(t/\tau)$ where $q_{30}(v) = 35v^4 - 84v^5 + 70v^6 - 20v^7$, $q_{31}(v) = -15v^4 + 39v^5 - 34v^6 + 10v^7$, $q_{32}(v) = \frac{15}{6}v^4 - 7v^5 + \frac{13}{2}v^6 - 2v^7$, $q_{33}(v) = -\frac{1}{6}v^4 + \frac{1}{2}v^5 - \frac{1}{2}v^6 + \frac{1}{6}v^7$. For example, choose $\tau = 2$ s and apply the inversion formula (39) to obtain $\tilde{u}_d(t) = 1.211e^t sin(t + 1.450)$, $t < 0$ (preaction control); $\tilde{u}_d(t) = 0.011458t^7 + 0.44010t^6 + 5.2062t^5 + 22.455t^4 + 25.781t^3 - 71.470t^2 - 202.78t - 132.65 + 1.7900e^{-t} + 148.41e^t \sin(t + 1.0835)$, $t \in [0, \tau]$; $\tilde{u}_d(t) = 0.9618\sin(2t + 3.506) + 1.160 \cdot 10^{-6}e^{-t}$, $t > \tau$ (steady-state input plus postaction control). Fig. 3 plots the resulting solution.

To obtain a continuous input, a smoothness degree 4 can be chosen for $\tilde{y}_d$. Still using $\tau = 2$ s, the corresponding interpolating polynomial is $p(t) = -\frac{65}{384}t^9 + \frac{99}{64}t^8 - \frac{43}{8}t^7 + \frac{101}{12}t^6 - 5t^5$ (details on how to compute it can be found in Costalunga and Piazzi (2015)).

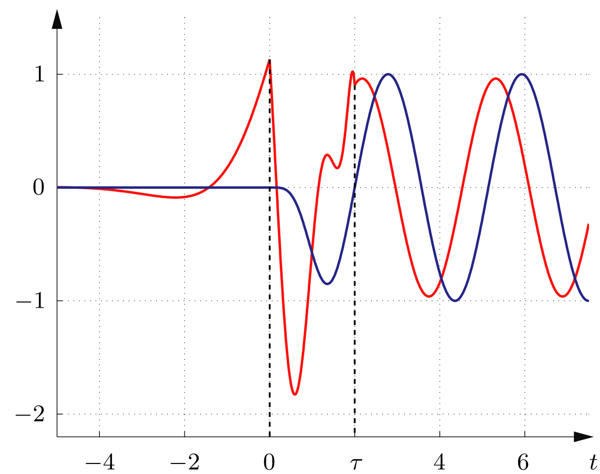

Fig. 4. Example 2: the output $\tilde{y}_d$ with smoothness degree 4 (in blue) and the inverse input $\tilde{u}_d$ (in red). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Apply (39) and obtain $\tilde{u}_d(t) = 1.143e^t \sin(t + 1.430)$, $t < 0$ (preaction control); $\tilde{u}_d(t) = -0.00846354t^9 - 0.417773t^8 - 6.60625t^7 - 40.5510t^6 - 78.2965t^5 + 272.146t^4 + 1744.78t^3 + 3680.90t^2 + 1939.80t - 1494.57 - 98.4656e^{-t} + 3972.53e^t \sin(t + 2.72866)$, $t \in [0, \tau]$; $\tilde{u}_d(t) = 0.9617 \sin(2t + 3.506) + 1.294 \cdot 10^{-5} e^{-t}$, $t > \tau$ (steady-state input plus postaction control). The resulting pair $(\tilde{u}_d, \tilde{y}_d)$ is plotted in Fig. 4.

5.1. Discussion

The presented examples show the relevance of *output (signal) design* in the inversion-based control. Besides the relative degree condition (38) that dictates the minimum smoothness degree of the output (equal to $r - 1$), the actual choice of this smoothness is relevant. Depending on the application, the control engineer can choose the output smoothness accordingly to the appropriate input smoothness (cf. Corollary 1). For example, a degree -1 of the input smoothness, i.e. a discontinuous input, can be fine for some applications whereas for some others a smoother input is more appropriate or even mandatory. This is highlighted in Example 1 where the hub angle input requires a second order continuity so that the smoothed output must have smoothness degree 2 or greater (cf. (46)).

Another aspect of output design regards the optimality of the input–output pair. The above examples make evident the opportunity to minimize the smoothing time τ in order to reduce the delay on the output of the occurrence of the desired output y_d. This can be done by setting an optimization problem with constraints on the amplitude of the (output) interpolating polynomial and its derivatives (as pursued in Costalunga & Piazzi, 2015) or with constraints on the amplitude of the inverse input and its derivatives (as done for set-point regulation problems in Piazzi and Visioli (2005)) or maybe with constraints on both the input and the output. More generally, the optimality issue in the inversion-based control has been addressed in Dewey, Leang, and Devasia (1998); Jetto, Orsini, and Romagnoli (2015); Perez and Devasia (2003).

In some cases, inversion-based control can be applied as a purely feedforward (open-loop) control (Bayo, 1987; Benosman & Le Vey, 2003; Piazzi & Visioli, 2000), but in the majority of control applications a feedforward–feedback implementation must be considered. Indeed, stable inversion is performed on a nominal system which may be quite different from the real system affected by uncertainties and discrepancies. For nonlinear systems, a scheme was presented in Devasia et al. (1996). The feedforward inverse input is computed by stable inversion on the nominal plant (the controlled system) and state feedback is introduced to stabilize the resulting plant trajectories. In the linear case, two schemes using output feedback controllers were proposed: (1) the *closed-loop inversion* architecture (Piazzi & Visioli, 2001b, d, 2002) and (2) the *plant inversion* architecture (Devasia, 2002; Peng, Xu, Zou, & Zhang, 2012; Wu & Zou, 2009). In the former, the feedforward input is computed by stable inversion on the closed-loop system with the feedback controller having the role to reduce the sensitivity of the closed-loop system to uncertainties, disturbances and unmodeled dynamics. In the latter, stable inversion on the plant determines the feedforward input with the feedback controller that reduces the tracking error between the desired output and the actual one. Both schemes can adhere to the *internal model principle* (Francis & Wonham, 1976) so that robust steady-state regulation can be achieved. Preliminary comparisons on these control architectures have been reported in Butterworth, Pao, and Abramovitch (2009).

6. Conclusion

Inversion-based control has been presented in a simplified behavioral framework for linear, nonminimum-phase SISO systems. To summarize, this approach has enabled us: (1) to find a straightforward solution to a general stable inversion problem; (2) to present a general formulation of the postaction control property; and (3) to emphasize the smoothness degree of the desired output as a key concept in addressing inversion-based control. By-products of the presented approach have been new results of behavior theory such as e.g. the characterization of weak solutions in terms of a differential–integral equation (with additional smoothness requirements) and the inverse representation of the behavior set. Future developments should focus on extending the present work to nonminimum-phase MIMO (multi-input multi-output) linear systems and on inversion-based feedforward–feedback control architectures.

Appendix. Proofs

Proof of Lemma 1. If $k = 0$ the statement is trivial. The proof is obtained by mathematical induction. First consider the base case $k = 1$. If $p = -1$ then $\int^1 f \in C^0$. If $p \geq 0$, $D^{p+1}(\int^1 f) = D^p(D(\int^1 f)) = D^p f \in C^0$, then $\int^1 f \in C^{p+1}$. Now suppose that (1) is true for a given $k \geq 1$. Then $D^{p+k+1}\left\{\int^{k+1} f\right\} = D^{p+k}\left\{D\left[\int_0^t \left(\int^k f\right)(\xi) d\xi\right]\right\} = D^{p+k}\left\{\int^k f\right\} \in C^0$. Hence $\int^{k+1} f \in C^{p+k+1}$ and this completes the proof. □

Proof of Lemma 2. Consider $f \in C_p^\infty$ which is defined on $\mathbb{R} \setminus S_f^{(0)}$. First prove that for $i \in \mathbb{N}$

$$D^i \int^i f = f \text{ on } \mathbb{R} \setminus S_f^{(0)}. \tag{A.1}$$

Identity (A.1) is trivial when $i = 0$. Using mathematical induction, for $i = 1$ we have $D\left(\int f\right)(t) = f(t)$, $t \in \mathbb{R} \setminus S_f^{(0)}$ because f is continuous in t. Now, assume (A.1) true for a given $i \geq 1$. Then $D^{i+1}\left(\int^{i+1} f\right) = D^i\left[D\left(\int\left(\int^i f\right)\right)\right] = D^i\left(\int^i f\right)$ on $\mathbb{R} \setminus S_f^{(0)}$. Using result (A.1), when $p > k$ the following identities hold on $\mathbb{R}$: $D^k \int^p f = D^k \int^k \left(\int^{p-k} f\right) = \int^{p-k} f$. When $k \geq p$ it follows that $D^k \int^p f = D^{k-p}\left(D^p \int^p\right) f = D^{k-p} f = \int^{p-k} f$ on $\mathbb{R} \setminus S_f^{(k-p)}$. □

Proof of Lemma 3. Without loss of generality, consider $t > 0$. If set $S_f^{(1)} \cap (0, t)$ is empty, relation (2) is just the fundamental theorem of calculus. Otherwise define $\{t_1, t_2, \ldots, t_k\} := S_f^{(1)} \cap (0, t)$, with $k \in \mathbb{N}^+$. Considering that $f \in C_p^\infty$ the following relations follow

$$\int_0^t Df(\xi)d\xi = \int_{0^+}^{t^-} Df(\xi)d\xi = \int_{0^+}^{t_1^-} Df(\xi)d\xi + \sum_{i=1}^{k-1} \int_{t_i^+}^{t_{i+1}^-} Df(\xi)d\xi + \int_{t_k^+}^{t^-} Df(\xi)d\xi$$
$$= f(t_1^-) - f(0^+) + \sum_{i=1}^{k-1} \left[f(t_{i+1}^-) - f(t_i^+)\right] + f(t^-) - f(t_k^+).$$

Hence, the continuity of f implies relation (2). $\square$

Proof of Lemma 4. First, consider $q = p$ and prove relation (3) by mathematical induction. Lemma 3 proves relation (3) when $p = 1$. Assume that (3) holds for $p \geq 1$. Then, consider $f \in C_p^\infty \cap C^p$ and the following passages

$$\int^{p+1} D^{p+1} f(t) = \int \left(\int^p D^p (Df)\right)(t) = \int \left[Df(t) - \sum_{i=0}^{p-1} \frac{f^{(i+1)}(0)}{i!} t^i\right]$$
$$= f(t) - f(0) - \sum_{i=0}^{p-1} \frac{f^{(i+1)}(0)}{i!} \int_0^t v^i dv$$
$$= f(t) - f(0) - \sum_{i=0}^{p-1} \frac{f^{(i+1)}(0)}{(i+1)!} t^{i+1}$$
$$= f(t) - \sum_{j=0}^{p} \frac{f^{(j)}(0)}{j!} t^j.$$

From the last equality we obtain relation (3) when $p + 1$ replaces p. Hence, the inductive step is completed.

Now consider $q > p$. Then

$$\int^q D^p f(t) = \int^{q-p} \int^p D^p f(t) = \int^{q-p} f(t) - \sum_{i=0}^{p-1} \frac{f^i(0)}{i!} \int^{q-p} t^i.$$

Since $\int^{q-p} t^i = \frac{i!}{(i+q-p)!} t^{i+q-p}$, relation (3) follows. $\square$

Proof of Theorem 1. Let us consider $A, B \in \mathcal{S}_\mathrm{i}$ (when $A, B \in \mathcal{S}_\mathrm{d}$ cf. (Apostol, 1969 p. 149)). Also let $A = \sum_{i=0}^n a_i \int^i$ and $B = \sum_{i=0}^m b_i \int^i$. From definition (4) the associated polynomials are $p_A(s) = \sum_{i=0}^n a_i s^i$ and $p_B(s) = \sum_{i=0}^m b_i s^i$.

(a) ($\Longrightarrow$) Choose the constant function $f(t) = 1$, $t \in \mathbb{R}$. Hence, $A(f)(t) = \sum_{i=0}^n (a_i/i!) t^i$ and $B(f)(t) = \sum_{i=0}^m (b_i/i!) t^i$. From $A = B$, it follows that $A(f)(t) = B(f)(t)$ for all $t \in \mathbb{R}$. This implies that $n = m$ and $a_i = b_i$, for $i = 0, 1, \ldots, n$, i.e. $p_A = p_B$.
($\Longleftarrow$) From $p_A = p_B$ it follows that $n = m$ and $a_i = b_i$, $i = 0, 1, \ldots, n$. Hence, for any $f \in C_p^\infty$ we have $A(f) = B(f)$, i.e. $A = B$.

(b) Given $f \in C_p^\infty$, $(A + B)f = Af + Bf = \sum_{i=0}^n a_i \int^i f + \sum_{i=0}^m b_i \int^i f = (\sum_{i=0}^n a_i \int^i + \sum_{i=0}^m b_i \int^i) f$. Hence $p_{A+B}(s) = \sum_{i=0}^n a_i s^i + \sum_{i=0}^m b_i s^i = p_A(s) + p_B(s)$.

(c) Given $f \in C_p^\infty$, $\lambda A(f) = \lambda(Af) = \lambda \sum_{i=0}^n a_i \int^i f = \sum_{i=0}^n \lambda a_i \int^i f$. Hence $p_{\lambda A}(s) = \sum_{i=0}^n \lambda a_i s^i = \lambda \sum_{i=0}^n a_i s^i = \lambda p_A(s)$.

(d) Given $f \in C_p^\infty$, $AB(f) = A(Bf) = \sum_{i=0}^n a_i \int^i (\sum_{j=0}^m b_j \int^j f) = \sum_{i=0}^n \sum_{j=0}^m a_i b_j \int^{i+j} f$. Hence $p_{AB}(s) = \sum_{i=0}^n \sum_{j=0}^m a_i b_j s^{i+j} = \sum_{i=0}^n a_i s^i (\sum_{j=0}^m b_j s^j) = p_A(s) \cdot p_B(s)$. $\square$

Proof of Theorem 2. Let $A = \sum_{i=-m_1}^{n_1} a_i \int^i$ and $B = \sum_{i=-m_2}^{n_2} b_i \int^i$. The associated Laurent polynomials are $p_A(s) = \sum_{i=-m_1}^{n_1} a_i s^i$ and $p_B(s) = \sum_{i=-m_2}^{n_2} b_i s^i$.

(a) ($\Longrightarrow$) Set $q := \max\{m_1, m_2\}$ and $f(t) := t^q$, $t \in \mathbb{R}$. Hence, $A(f)(t) = q! \sum_{i=-m_1}^{n_1} (a_i/(q+i)!) t^{q+i}$ and $B(f)(t) = q! \sum_{i=-m_2}^{n_2} (b_i/(q+i)!) t^{q+i}$. From $A(f)(t) = B(f)(t)$ for all $t \in \mathbb{R}$ it follows that $m_1 = m_2$, $n_1 = n_2$ and $a_i = b_i$, $i = -m_1, \ldots, n_1$, i.e. $p_A = p_B$.
($\Longleftarrow$) From $p_A = p_B$ it follows that $m_1 = m_2$, $n_1 = n_2$ and $a_i = b_i$, $i = -m_1, \ldots, n_1$. Hence, given any $f \in C_p^\infty$ then $A(f) = B(f)$, i.e. $A = B$.

(b) Given $f \in C_p^\infty$, $(A + B)f = Af + Bf = \sum_{i=-m_1}^{n_1} a_i \int^i f + \sum_{i=-m_2}^{n_2} b_i \int^i f = (\sum_{i=-m_1}^{n_1} a_i \int^i + \sum_{i=-m_2}^{n_2} b_i \int^i) f$. Hence $p_{A+B}(s) = \sum_{i=-m_1}^{n_1} a_i s^i + \sum_{i=-m_2}^{n_2} b_i s^i = p_A(s) + p_B(s)$.

(c) Given $f \in C_p^\infty$, $\lambda A(f) = \lambda(Af) = \lambda \sum_{i=-m_1}^{n_1} a_i \int^i f = \sum_{i=-m_1}^{n_1} \lambda a_i \int^i f$. Hence $p_{\lambda A}(s) = \sum_{i=-m_1}^{n_1} \lambda a_i s^i = \lambda \sum_{i=-m_1}^{n_1} a_i s^i = \lambda p_A(s)$. $\square$

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