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Inverse Feedforward Control with Output Polynomial Smoothing

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***Abstract*—A method of polynomial smoothing is presented for the inverse feedforward control of continuous-time, linear, nonminimum-phase SISO (single-input single-output) systems. Starting from an arbitrarily given output signal, a smoothing procedure is devised to obtain a delayed smoothed output for which stable input-output inversion can be applied. This procedure requires to solve a polynomial interpolating problem whose solution is given by a polynomial parameterized by the smoothing time parameter. Detailed expressions of the deduced inverse input are provided. The minimization of the smoothing time is also pursued in order to reduce the delay in the smoothed output. Examples highlight the proposed method. The aim of the presented approach is to achieve high-performances in a variety of possible control applications such as, e.g., those in process automation and mechatronics.**

***Keywords*—feedforward control; stable inversion; nonminimum-phase systems; linear systems; polynomial smoothing.**

I. Introduction

In recent times there has been a growing interest in feedforward control methods [1]. Indeed, they complement the feedback control methods and can be applied to achieve high-performances in a variety of applications such as, e.g., those in process automation, mechatronics, industrial and autonomous robotics, etc. [2], [3], [4]. Among the various methods (bang-bang control, input shaping techniques, etc.) inversion-based feedforward control methods have found their way in mid 90's and subsequent years [5], [6], [7], [8]. These methods share a common idea. First, an output signal is designed according to the pertinent application. Then, by system inversion, the corresponding (inverse) input that causes the desired output is determined.

In this approach, a difficulty was found in the application to nonminimum-phase systems, i.e., systems whose zero dynamics is unstable [9]. Indeed, for these systems the standard inversion procedure fails to provide an acceptable solution insofar the obtained input is unbounded even in presence of a bounded desired output signal. This theoretical obstruction was overcome by the works in [10], [11], [5], [6]. The idea that led to the breakthrough was to extend the space of solutions to noncausal signals. In such a way, it emerged a line of research devoted to *noncausal stable inversion*. In this line, one of the first addressed problems was that of feedforward regulation, i.e., the problem to cause a given system to make an output transition between a current value to a future (constant) value [7], [12], [13]. In particular, in [13] *transition polynomials* were used to smoothly shape a monotonically increasing output signal between the current and future output values.

In this work, still in the context of continuous-time, linear, nonminimum-phase SISO systems, we extend the results of [13] by addressing and solving a more general control problem. Specifically, we consider an arbitrary desired causal output signal which is not necessarily bounded and provide a polynomial smoothing of this signal in order to apply the stable input-output inversion [14]. Apart very special cases, an output smoothing is necessary because a relative degree condition (cf. Proposition 3) must be satisfied and as an appropriate and convenient choice, a polynomial interpolation problem is posed and solved (cf. Problem 3).

Paper organization: Section II provides the key definitions of *polynomial order* and *smoothness degree* of a signal, succinctly introduces the stable inversion, and gives motivation for posing a kth-order interpolation problem. A polynomial solution to this problem is presented in Section III. The structure of the smoothing polynomial is given by Proposition 5. Section IV reports detailed expressions of the inverse input and the pertinent associated problem of minimizing the smoothing time (Problem 4). In Section V, two examples highlight the proposed inverse feedforward control. A brief conclusion ends the paper in Section VI.

Notation: We say that a real function $f : \mathbb{R} \to \mathbb{R}$ has continuity order n if it belongs to C^n, the set of continuous functions with continuous derivatives till the nth-order. The nth-order derivative of a real function f is denoted by $f^{(n)}$ or $D^n f$. A function f belongs to PC^∞, the set of piecewise C^∞-functions if there exists a numerable set $\{t_1, t_2, \dots\}$ (it may be the empty set) such that $f \in C^\infty(\mathbb{R}\backslash\{t_1, t_2, \dots\})$ and there exist finite limits $\lim_{t\to t_i^-} f^{(k)}$, $\lim_{t\to t_i^+} f^{(k)}$ for all $k \in \mathbb{N}$ and $t_i \in \{t_1, t_2, \dots\}$. As an extension of the C^n sets ($n \in \mathbb{N}$), define $C^{-1} := PC^\infty$.
A function f is said to be *causal* if $f(t) = 0$ for all $t < 0$. The Heaviside function is denoted by $1(t)$ according to the definition: $1(t) = 1$ it $t \geq 0$ and $1(t) = 0$ if $t < 0$. A polynomial $p(s)$ is said to be Hurwitz if all its roots have negative real parts.

II. Stable input-output inversion and problem motivation

Let us consider a linear time-invariant system Σ with transfer function $H(s)$ given by:

$$H(s) = \frac{b(s)}{a(s)} = \frac{b_m s^m + b_{m-1}s^{m-1} + \dots + b_0}{a_n s^n + a_{n-1}s^{n-1} + \dots + a_0}.$$

We assume that $H(s)$ is a proper rational function, polynomials $a(s)$ and $b(s)$ are coprime, and the zeros of Σ do not belong to the imaginary axis of the complex plane, i.e, the roots of $b(s)$ have real part which is positive or negative. The order of Σ is n and its relative degree is $r := n - m$. The input and output of Σ are $u \in \mathbb{R}$ and $y \in \mathbb{R}$ respectively. The behavior of Σ, i.e, the set of all pairs of input and corresponding output signals, can be introduced as follows [15]. Consider the differential

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equation associated to system Σ:

$$\sum_{i=0}^{n} a_i D^i y(t) = \sum_{i=0}^{m} b_i D^i u(t); \tag{1}$$

then the behavior of Σ can be defined as:

$$\mathcal{B} := \{\ (u, y) \in PC^\infty \times PC^\infty : \\ (u, y) \text{ is a weak solution of (1) } \}. \tag{2}$$

Pair (u, y) is a weak solution in $\mathcal{B}$ if there exists a sequence $(u_k, y_k) \in C^\infty \times C^\infty, k \in \mathbb{N}$, satisfying differential equation (1) such that the (u_k, y_k) converge to (u, y) as k goes to infinity. A complete theory on system behaviors is reported in Polderman and Willems' book [15].

A definition which is relevant to our approach to inversion control is the following.

***Definition** 1 (Polynomial order of a signal):* A signal $f \in PC^\infty$ has polynomial order l if there exist constants $M > 0$ and $N > 0$ such that

$$|f(t)| < M\, t^l + N, \forall t \geq 0. \tag{3}$$

The stable input-output inversion problem can be introduced as follows.

***Problem** 1 (Stable inversion problem):* Let be given a sufficiently smooth causal signal $y \in PC^\infty$. Assume that y and its derivatives $Dy, \ldots, D^r y$ have all polynomial order l. Find an (inverse) signal $u \in PC^\infty$ with polynomial order l such that $(u, y) \in \mathcal{B}$.

Consider causal input and output signals u and y, such that $(u, y) \in \mathcal{B}$. Their Laplace transforms satisfy the relation $Y(s) = H(s)U(s)$. By Euclidean division $a(s)$ can be expressed as $a(s) = c(s)b(s) + d(s)$, with

$$c(s) = c_r s^r + c_{r-1} s^{r-1} + \cdots + c_0,\ c_r = \frac{a_n}{b_m} \neq 0 \tag{4}$$

and $\deg d(s) < m$. Hence $U(s) = c(s)Y(s) + H_0(s)Y(s)$ where $H_0(s) = \frac{d(s)}{b(s)}$ represents the zero dynamics of Σ ([9]). Let $h_0(t) = \mathcal{L}^{-1}[H_0(s)]$ and by Laplace inverse transform we obtain

$$u(t) = c_r D^r y(t) + \cdots + c_0 y(t) + \int_0^t h_0(t - v) y(v) dv,\ t \in \mathbb{R} \tag{5}$$

which is the standard inverse for which $(u, y) \in \mathcal{B}$.

If the zero dynamics is stable, i.e., $b(s)$ is an Hurwitz polynomial, the above expression is the solution of the stable inversion (Problem 1). However, in the case of an unstable zero dynamics (i.e. of nonminimum-phase systems) the inverse input (5) still satisfies $(u, y) \in \mathcal{B}$ but it does not possess any polynomial order because it diverges exponentially (cf [5]). From early works appearing in [10], [11], [5], [6], it is known that the stable inversion problem when the given y belongs to the Lebesgue spaces $L_1 \cap L_\infty$ or is a polynomial still admits a solution as far as noncausal inputs are admitted.

The solution of Problem 1 can be presented as follows [14]. The zero polynomial $b(s)$ can be factored as $b(s) = b_m\, b^-(s)\, b^+(s)$ with $b^-(s)$ and $b^+(s)$ being monic polynomials that have all there roots with negative and positive real part respectively (i.e. $b^-(s)$ and $b^+(-s)$ are Hurwitz polynomials). Hence

$$H_0(s) = \frac{d(s)}{b_m\, b^-(s)\, b^+(s)} \tag{6}$$

and by partial fraction decomposition the zero dynamics of Σ is decomposed as the sum of its stable and unstable parts:

$$H_0(s) = H_0^-(s) + H_0^+(s), \tag{7}$$

$$H_0^-(s) := \frac{d^-(s)}{b^-(s)},\ H_0^+(s) := \frac{d^+(s)}{b^+(s)} \tag{8}$$

with $d^-(s)$,$d^+(s)$ being suitable polynomials. Denote by $h_0^-(t)$ and $h_0^+(t)$ the analytic extension over $\mathbb{R}$ of $\mathcal{L}^{-1}[H_0^-(s)]$ and $\mathcal{L}^{-1}[H_0^+(s)]$ respectively.

***Proposition** 1 ([14]):* The unique solution of the stable inversion problem (Problem 1) is given by

$$u(t) = c_r D^r y(t) + \cdots + c_0 y(t) \\ + \int_0^t h_0^-(t - v) y(v) dv - \int_t^\infty h_0^+(t - v) y(v) dv,\ t \in \mathbb{R}. \tag{9}$$

A property of input-output pairs of the behavior $\mathcal{B}$ which is relevant for the inversion-based feedforward control is the following [15], [16].

***Proposition** 2:* Consider a pair $(u, y) \in \mathcal{B}$. Let $p \in \mathbb{Z}$ with $p \geq -1$. Then $u \in C^p$ if and only if $y \in C^{r+p}$.

A useful definition is the following.

***Definition** 2 (Smoothness degree):* A signal $f \in PC^\infty$ is said to have smoothness degree $k \geq -1$ if $f \in C^k$ and $D^{k+1} f$ has jump discontinuities, i.e. there exists at least one $t_j \in \mathbb{R}$ such that $\lim_{t \to t_j^-} D^{k+1} f(t) \neq \lim_{t \to t_j^+} D^{k+1} f(t)$. Signal f has *infinite* smoothness. i.e. $k = \infty$ when $f \in C^\infty$.

In a way, the above definition of smoothness degree corresponds to the concept of maximal continuity order of a signal. Proposition 2 can be then restated in terms of smoothness degrees of the input and output signals.

***Corollary** 1:* Consider a pair $(u, y) \in \mathcal{B}$. Then, input u has smoothness degree k if and only if y has smoothness degree $k + r$.

The inverse feedforward control problem can be then posed as follows:

***Problem** 2 (Inverse feedforward control problem):* Let be given a desired causal output signal $y_d \in PC^\infty$ with smoothness degree k. Assume that y_d and its derivatives $Dy_d, \ldots, D^r y_d$ have all polynomial order l. Find an (inverse) input $u_d \in PC^\infty$ with polynomial order l such that $(u_d, y_d) \in \mathcal{B}$.

By virtue of Proposition 1 and Corollary 1 the following result can be established.

***Proposition** 3 (Relative degree condition):* The inverse feedforward control problem (Problem 2) has a solution if and only if

$$k \geq r - 1, \tag{10}$$

i.e. the smoothness degree of the desired output signal y_d is greater or equal to the relative degree of Σ minus one.

Proof of proposition 3 is omitted for brevity. It is worth noting that the inverse input, solution of Problem 2, is still given by expression (9). However, the exact smoothness degree required on the desired output for which (9) expresses the valid input (i.e. that causes the desired output) must satisfy the relative degree condition (10). In a control engineering context, in many cases this condition can impede the direct application of the inversion-based feedforward control. Indeed, usually y_d is an analytic function over $[0, +\infty)$ (and hence $y_d \in C^\infty([0, +\infty))$) but the discontinuity at time zero can disrupt the smoothness of y_d over $\mathbb{R}$ requested by condition (10). For example, when $y_d(t) = y_c\, 1(t)$ with y_c being a given constant, as it is customary in regulation problems, and $r \geq 1$, it is impossible to directly apply the inverse control.

A way to overcome this theoretical obstruction is to resort to a suitable smoothing on the desired y_d. Introduce a smoothed output $\tilde{y}(t)$ according to

$$\tilde{y}(t) = \begin{cases} 0 & t < 0 \\ p(t) & t \in [0, \tau] \\ y_d(t - \tau) & t > \tau \end{cases} \tag{11}$$

where $\tau > 0$ is the time duration of the smoothing and $p(t)$ is an interpolating function to be designed to ensure $\tilde{y} \in C^k$, with $k \geq r - 1$. With this aim, we introduce the following interpolation problem.

***Problem** 3 (The kth-order interpolation problem):* Assume $y_d \in C^k([0, \infty))$. Find a function $p(t) \in C^k([0, \tau])$ such that the following interpolating conditions are satisfied at the endpoints of interval $[0, \tau]$:

$$p(0) = 0,\, p^{(1)}(0) = 0, \ldots, p^{(k)}(0) = 0 \tag{12}$$

$$p(\tau) = y^{(0)},\, p^{(1)}(\tau) = y^{(1)}, \ldots, p^{(k)}(\tau) = y^{(k)} \tag{13}$$

where

$$y^{(i)} := y_d^{(i)}(0),\, i = 0, \ldots, k. \tag{14}$$

Clearly, the above definitions (14) with conditions (12), (13) imply that the resulting $\tilde{y}$ actually belongs to $C^k(\mathbb{R})$. In the next section we propose a polynomial solution to the above interpolating problem.

III. Output polynomial smoothing

To satisfy the $2k+2$ interpolating conditions given by (12) and (13) consider a $(2k+1)$-order polynomial with parameters p_0, p_1, …, p_{2k+1} as follows

$$p(t) = p_0 + p_1 \left(\frac{t}{\tau}\right) + p_2 \left(\frac{t}{\tau}\right)^2 + \cdots + p_{2k+1} \left(\frac{t}{\tau}\right)^{2k+1}. \tag{15}$$

From interpolation conditions (12) we immediately get $p_0 = 0$, $p_1 = 0$, …, $p_k = 0$, so that

$$p(t) = p_{k+1} \left(\frac{t}{\tau}\right)^{k+1} + p_{k+2} \left(\frac{t}{\tau}\right)^{k+2} + \cdots + p_{2k+1} \left(\frac{t}{\tau}\right)^{2k+1}. \tag{16}$$

By computing the derivatives of $p(t)$ till to the k-th order and by imposing conditions (13) the following system of linear equations is obtained

$$\begin{cases} p_{k+1} + p_{k+2} + \cdots + p_{2k+1} = y^{(0)} \\ \frac{(k+1)}{\tau} p_{k+1} + \frac{(k+2)}{\tau} p_{k+2} + \cdots + \frac{(2k+1)}{\tau} p_{2k+1} = y^{(1)} \\ \vdots \\ \frac{(k+1)!}{\tau^k} p_{k+1} + \frac{(k+2)!}{2!\,\tau^k} p_{k+2} + \cdots + \frac{(2k+1)!}{(k+1)!\,\tau^k} p_{2k+1} = y^{(k)} \end{cases}. \tag{17}$$

By multiplying the above equations, from the second to the last one, for τ^i, $i = 1, \ldots, k$ the equivalent system follows:

$$\begin{cases} p_{k+1} + \cdots + p_{2k+1} = y^{(0)} \\ (k+1) p_{k+1} + \cdots + (2k+1) p_{2k+1} = \tau y^{(1)} \\ \vdots \\ (k+1)! p_{k+1} + \cdots + \frac{(2k+1)!}{(k+1)!} p_{2k+1} = \tau^k y^{(k)} \end{cases}. \tag{18}$$

The coefficient matrix of system (18) is denoted by

$$M(k) := \begin{bmatrix} 1 & 1 & \ldots & 1 \\ k+1 & k+2 & \ldots & 2k+1 \\ \vdots & & & \\ (k+1)! & \frac{(k+2)!}{2!} & \ldots & \frac{(2k+1)!}{(k+1)!} \end{bmatrix} \tag{19}$$

so that (18) is rewritten as

$$M(k) \begin{bmatrix} p_{k+1} \\ p_{k+2} \\ \vdots \\ p_{2k+1} \end{bmatrix} = \begin{bmatrix} y^{(0)} \\ \tau\, y^{(1)} \\ \vdots \\ \tau^k\, y^{(k)} \end{bmatrix}. \tag{20}$$

The following result is crucial in solving (20).

***Proposition** 4:* Matrix $M(k)$ is nonsingular for all $k \geq 0$.

Proof: We use a contradiction argument. By absurd, suppose there exists $\bar{k} \geq 0$ for which $M(\bar{k})$ is singular. Choose $\tau = 1$ and fix to zero all the interpolation data: $y^{(0)} = 0$, $y^{(1)} = 0$, …, $y^{(\bar{k})} = 0$. Hence, there exist coefficients $p_{\bar{k}+1}$, $p_{\bar{k}+2}$, …, $p_{2\bar{k}+1}$ not all zero, solution of system (20) so that the nonzero polynomial

$$p(t) = p_{\bar{k}+1}\, t^{\bar{k}+1} + p_{\bar{k}+2}\, t^{\bar{k}+2} + \cdots + p_{2\bar{k}+1}\, t^{2\bar{k}+1} \tag{21}$$

satisfies

$$p(0) = 0,\, p^{(1)}(0) = 0, \ldots, p^{(\bar{k})}(0) = 0, \tag{22}$$

$$p(1) = 0,\, p^{(1)}(1) = 0, \ldots, p^{(\bar{k})}(1) = 0. \tag{23}$$

Equalities (22), (23) imply that $t^{\bar{k}+1}$ and $(t-1)^{\bar{k}+1}$ are factors of $p(t)$ so that $\deg p \geq 2\bar{k} + 2$. This inequality is contradicted by expression (21) for which $\deg p \leq 2\bar{k} + 1$. ■

The matrix inverse $M^{-1}(k)$ be denoted by $N(k) \in \mathbb{Q}^{(k+1)\times(k+1)}$, so that the unique solution of (20) can be written as

$$\begin{bmatrix} p_{k+1} \\ p_{k+2} \\ \vdots \\ p_{2k+1} \end{bmatrix} = N(k) \begin{bmatrix} y^{(0)} \\ \tau y^{(1)} \\ \vdots \\ \tau^k y^{(k)} \end{bmatrix} \tag{24}$$

and explicitly as

$$\begin{cases} p_{k+1} = N_{1,1}(k)y^{(0)} + \cdots + N_{1,k+1}(k)\tau^k y^{(k)} \\ p_{k+2} = N_{2,1}(k)y^{(0)} + \cdots + N_{2,k+1}(k)\tau^k y^{(k)} \\ \vdots \\ p_{2k+1} = N_{k+1,1}(k)y^{(0)} + \cdots + N_{k+1,k+1}(k)\tau^k y^{(k)} \end{cases} . \tag{25}$$

By using (25) and rearranging the addends in the expression (16) we can finally obtain this result.

***Proposition** 5 (The smoothing polynomial):* There exists a unique solution to the kth-order interpolation problem (12)-(14) in the set of $(2k+1)$-order polynomials (cf. (15)). This can be expressed as

$$p(t) = y^{(0)}q_{k0}(t/\tau) + y^{(1)}\tau q_{k1}(t/\tau) + \cdots + y^{(k)}\tau^k q_{kk}(t/\tau) \tag{26}$$

where, $i = 0, 1, \ldots, k$,

$$q_{ki}(v) := N_{1,i+1}(k)v^{k+1} + \cdots + N_{k+1,i+1}(k)v^{2k+1}. \tag{27}$$

Computation of polynomials $q_{ki}(v)$ can be performed by means of a computer algebra system. For the special case of $y^{(0)} = 1$ and $y^{(1)} = 0, \ldots, y^{(k)} = 0$, polynomial (26) becomes the *transition polynomial* introduced in [16], [13] for the inversion-based feedforward regulation of linear scalar systems. In particular, the work in [16] provides a closed-form expression of the transition polynomial as

$$p(t) = \frac{(2k+1)!}{k!\tau^{2k+1}} \sum_{i=0}^{k} \frac{(-1)^{k-i}}{i!(k-i)!(2k-i+1)} \tau^i t^{2k-i+1} \tag{28}$$

that can be rewritten as

$$p(t) = \frac{(2k+1)!}{k!} \sum_{i=k+1}^{2k+1} \frac{(-1)^{i-k-1}}{(i-k-1)!(2k-i+1)!i} \left(\frac{t}{\tau}\right)^i . \tag{29}$$

Hence, by comparing (29) with (26), the following closed-form expression of $q_{k0}(v)$ is obtained.

$$q_{k0}(v) = \frac{(2k+1)!}{k!} \sum_{i=k+1}^{2k+1} \frac{(-1)^{i-k-1}}{(i-k-1)!(2k-i+1)!i} v^i . \tag{30}$$

IV. Inverse feedforward control

Having solved the kth-order interpolation problem with the smoothing polynomial (26) (cf. Problem 3), the stable inversion formula (9) can be applied to the smoothed desired output signal $\tilde{y}(t)$ (cf. 11). Hence, detailed expressions of the (stable) inverse input $\tilde{u}(t)$ can be deduced in each of the relevant time intervals: when $t < 0$

$$\tilde{u}(t) = -\int_0^\tau h_0^+(t-v)p(v)dv - \int_\tau^{+\infty} h_0^+(t-v)y_d(v-\tau)dv; \tag{31}$$

if $t \in [0, \tau]$

$$\begin{aligned} \tilde{u}(t) = {} & c_r p^{(r)}(t) + \cdots + c_0 p(t) + \int_0^t h_0^-(t-v)p(v)dv \\ & - \int_t^\tau h_0^+(t-v)p(v)dv - \int_\tau^{+\infty} h_0^+(t-v)y_d(v-\tau)dv; \end{aligned} \tag{32}$$

and, finally, with $t > \tau$

$$\begin{aligned} \tilde{u}(t) = {} & c_r y_d^{(r)}(t-\tau) + \cdots + c_0 y_d(t-\tau) \\ & + \int_0^\tau h_0^-(t-v)p(v)dv + \int_\tau^t h_0^-(t-v)y_d(v-\tau)dv \\ & - \int_\tau^{+\infty} h_0^+(t-v)y_d(v-\tau)dv. \end{aligned} \tag{33}$$

Expression (31) gives the so-called *preaction control* (cf. [17]) which is typical in the inversion-based control of nonminimum-phase systems. Denote by $m_i^+(t)$, $i = 1, \ldots, m^+$ with $m^+ := \deg b^+(s)$ the function modes associated to the unstable zero dynamics [14]. Then, by evaluation of the integrals appearing in (31), the preaction control can be written as

$$\tilde{u}(t) = \sum_{i=1}^{m^+} g_i m_i^+(t) \tag{34}$$

where g_i, $i = 1, \ldots, m^+$ are suitable real coefficients. Preaction control is not identically zero over $(-\infty, 0)$ and has $\tilde{u}(t) \to 0$ as $t \to -\infty$. Hence in a practical application we need to truncate $\tilde{u}(t)$, i.e., to force $\tilde{u}(t) = 0$ whenever $t < t_p < 0$, where $|t_p|$ is the preactuation time, leading to an approximate inverse input [12].

The smoothing time τ delays the occurrence on the output of the desired y_d (cf. Problem 2). Therefore it is sensible to minimize this smoothing time . This can be done by solving the following optimization problem.

***Problem** 4 (Minimization of the smoothing time):*

$$\min_{\tau > 0} \tau \tag{35}$$

such that

$$|p^{(i)}(t)| \le p_{\text{ub}}^{(i)}, \forall t \in [0, \tau], i = 0, 1, \ldots, k+1 \tag{36}$$

where $p_{\text{ub}}^{(i)}$ are selectable bounds to be chosen according to

$$p_{\text{ub}}^{(i)} \ge |y^{(i)}|, \; i = 0, 1, \ldots, k \tag{37}$$

and

$$p_{\text{ub}}^{(k+1)} > 0. \tag{38}$$

Obviously inequalities (37) and (38) are necessary conditions in order Problem 4 has a solution. In general, however, these inequalities do not make a sufficient condition. Hence, some care must be paid in choosing the bounds $p_{\text{ub}}^{(i)}$. Problem 4 is evidently equivalent to the following one:

$$\min \left\{ \tau > 0 : \max_{0 \le t \le \tau} |p^{(i)}(t)| \le p_{\text{ub}}^{(i)}, i = 0, 1, \ldots, k+1 \right\}. \tag{39}$$

Taking into account that $p(t)$ is a polynomial, for a given τ the maxima appearing in (39) can be easily determined. Hence, a standard local optimization routine to compute the solution τ^* of Problem 4 can be used. On the other hand, to obtain a guaranteed global solution, global optimization methods such as, eg, those based on interval analysis, could be adopted (cf. [18]). For the special case of $y^{(0)} \neq 0$ and $y^{(1)} = 0, \ldots, y^{(k)} = 0$ and $p_{\text{ub}}^{(k+1)} = +\infty$ (i.e. there is no bound on the $(k+1)$-order derivative) a closed form expression that gives the global solution τ^* has been presented in [16].

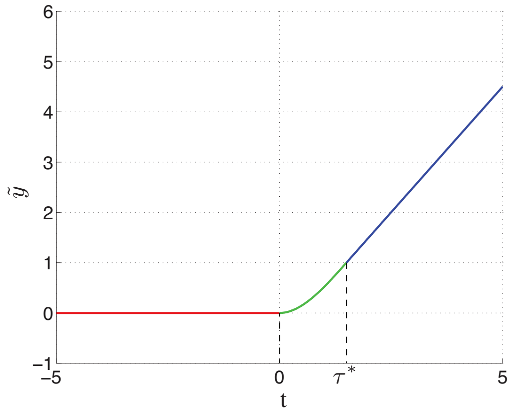

Fig. 1. The smoothed output $\tilde{y}$ for the first example: the green line plots the smoothing polynomial $p(t)$, $t \in [0, \tau^*]$ and the blue one plots the τ^*-delayed desired output $y_d(t - \tau^*)$.

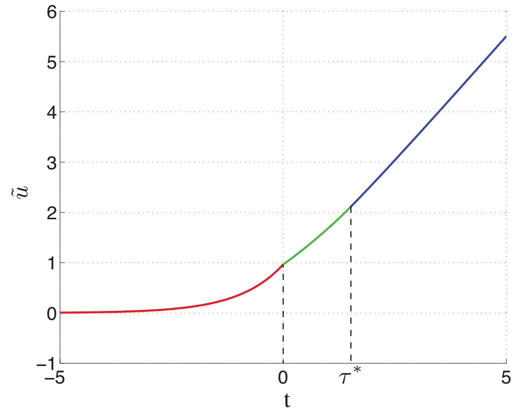

Fig. 2. The inverse input for the first example: the red line plots the preaction control (cf. (44)).

V. Examples

The first example uses a nonminimum-phase system taken from [13]. Its transfer function is

$$H_1(s) = -4\frac{(s+1)(s-1)}{(s+2)(s^2+s+2)}. \tag{40}$$

The desired output is $y_d(t) = (1+t)\,1(t)$ which is a step plus a ramp function. It is a discontinuous function (-1 is its smoothness degree) that does not satisfy the condition (10) of Proposition 3 because the relative degree of (40) is $r = 1$. Hence, a first order smoothing ($k = 1$) is proposed with the aim to obtain a continuous inverse input (cf. (14)):

$$y^{(0)} = 1,\ y^{(1)} = 1.$$

By Proposition 5, the smoothing polynomial is

$$p(t) = y^{(0)} q_{10}(t/\tau) + y^{(1)} \tau q_{11}(t/\tau) \tag{41}$$

where

$$q_{10}(v) = 3v^2 - 2v^3, \quad q_{11}(v) = -v^2 + v^3. \tag{42}$$

An optimal smoothing is posed with (cf. Problem 4)

$$p_{ub}^{(0)} = 1,\ p_{ub}^{(1)} = 1,\ p_{ub}^{(2)} = 2. \tag{43}$$

The corresponding minimum smoothing time is $\tau^* = 1.500$s. The smoothed output $\tilde{y}$ (cf. (11)) is plotted in Figure 1. By means of the inversion formulae (31)-(33) the inverse input $\tilde{u}$ can be determined as follows. If $t < 0$

$$\tilde{u}(t) = 0.9642 e^{t} \text{ (preaction control)}, \tag{44}$$

if $t \in [0, \tau^*]$

$$\tilde{u}(t) = \frac{4}{3} e^{t-\frac{3}{2}} - \frac{5}{9} e^{-t} - \frac{2}{9} t + \frac{2}{9} t^2 - \frac{4}{27} t^3 + \frac{11}{9} \tag{45}$$

and, finally, when $t > \tau^*$

$$\tilde{u}(t) = t + \frac{1}{2} + 0.4404 e^{-t}. \tag{46}$$

Figure 2 depicts the feedforward inverse input $\tilde{u}$. Note that, when $t > \tau^*$, the input $\tilde{u}$ can be written as

$$\tilde{u}(t) = u_{ss}(t) + 0.4404 e^{-t} \tag{47}$$

where $u_{ss}(t) = t + \frac{1}{2}$ is the steady-state input whereas $0.4404 e^{-t}$ is the decaying postaction control (cf. [19]).

As a second example, consider a system with the following transfer function

$$H_2(s) = 4\frac{(s+1)[(s-1)^2+1]}{(s+2)^5\left(s+\frac{1}{2}\right)^2}. \tag{48}$$

The desired output is $y_d(t) = \sin(2\,t)\,1(t)$ which has 0 as degree of smoothness (cf. Definition 2). Being the relative degree of the transfer function (48) equal to 4, Proposition 3 is not satisfied. Therefore, a fourth order smoothing ($k = 4$) is proposed:

$$y^{(0)} = 0,\ y^{(1)} = 1,\ y^{(2)} = 0,\ y^{(3)} = -1,\ y^{(4)} = 0.$$

Then, from Proposition 5, the smoothing polynomial is

$$\begin{aligned} p(t) = &\, y^{(0)} q_{40}(t/\tau) + y^{(1)} \tau q_{41}(t/\tau) + y^{(2)} \tau q_{42}(t/\tau) \\ &+ y^{(3)} \tau q_{43}(t/\tau) + y^{(4)} \tau q_{44}(t/\tau) \end{aligned}$$

with

$$\begin{aligned} q_{40}(v) &= 126v^5 - 420v^6 + 540v^7 - 315v^8 + 70v^9, \\ q_{41}(v) &= -56v^5 + 196v^6 - 260v^7 + 155v^8 - 35v^9, \\ q_{42}(v) &= \frac{21}{2}v^5 - \frac{77}{2}v^6 + 53v^7 - \frac{65}{2}v^8 + \frac{15}{2}v^9, \\ q_{43}(v) &= -v^5 + \frac{23}{6}v^6 - \frac{11}{2}v^7 + \frac{21}{6}v^8 - \frac{5}{6}v^9, \\ q_{44}(v) &= \frac{1}{24}v^5 - \frac{1}{6}v^6 + \frac{1}{4}v^7 - \frac{1}{6}v^8 + \frac{1}{24}v^9. \end{aligned}$$

An optimal smoothing is posed with bounds

$$\begin{aligned} p_{ub}^{(0)} = 1,\ p_{ub}^{(1)} = 2,\ p_{ub}^{(2)} = 20, \\ p_{ub}^{(3)} = 50,\ p_{ub}^{(4)} = 100,\ p_{ub}^{(5)} = 200. \end{aligned}$$

The corresponding minimum smoothing time is $\tau^* = 2.479$s. Figure 3 depicts the smoothed output $\tilde{y}$. The input $\tilde{u}$, which is obtained by the stable inversion formulae (31)-(33), is shown in Figure 4. Its closed-form expressions are:

$$\tilde{u}(t) = 23.94 \sin(t + 1.041) e^{t},\ t < 0 \text{ (preaction control)};$$

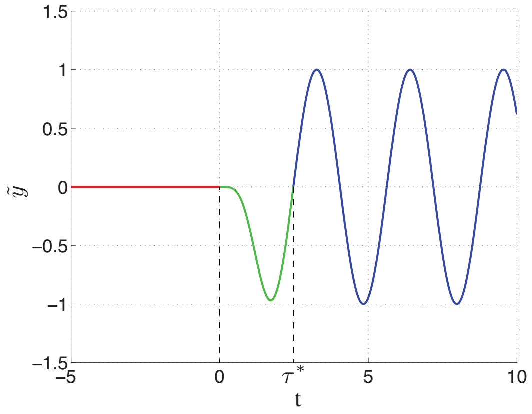

Fig. 3. The smoothed output $\tilde{y}$ for the second example: the green line plots the smoothing polynomial $p(t)$, $t \in [0, \tau^*]$ and the blue one plots the τ^*-delayed desired output $y_d(t - \tau^*)$.

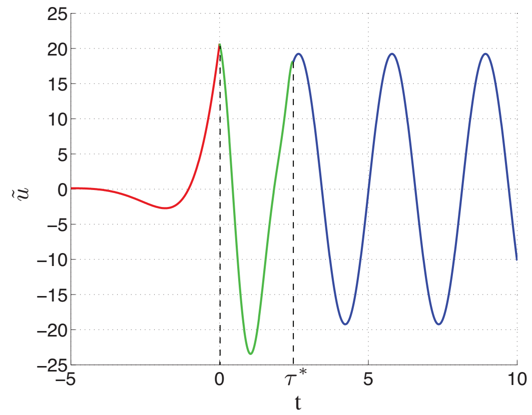

Fig. 4. The inverse input for the second example: the red line plots the preaction control.

$$\begin{aligned}\tilde{u}(t) = &- 7035.49 \sin(t - 0.797)e^{t} - 303.06e^{-t}\\ &- 0.02036t^9 - 0.9513t^8 - 13.52t^7\\ &- 64.76t^6 - 17.23t^5 + 906.01t^4\\ &+ 3301.93t^3 + 4976.37t^2 - 436.05t\\ &- 4708.67,\ t \in [0, \tau^*];\end{aligned}$$

$$\tilde{u}(t) = -19.23 \sin(2t - 0.5927) - 0.01370e^{-t},\ t > \tau^*.$$

Note that when $t > \tau^*$, the inverse input is the sum of a steady-state harmonic signal and a decaying postaction control.

VI. Conclusion

A new method of inverse feedforward control has been presented. It is based on the use of a smoothing polynomial whose time parameter τ can be minimized in order to reduce the delay in the smoothed desired output. The method can be also applied to the broader class of signals with finite polynomial order, i.e., signals that can grow in time with a polynomial law. Possible applications can be found in the high-performance control of automation processes and mechatronic devices.

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