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# Time-optimal dynamic path inversion for an automatic guided vehicle

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***Abstract*— The article presents the time-optimal trajectory planning of an automatic guided vehicle (AGV) on a given feasible path while respecting velocity, acceleration and jerk constraints. A theoretical result shows the connection for the AGV between the geometric continuity of its paths and the smoothness of its control inputs (linear velocity and steering angle of the AGV motor wheel). The solution hence proposed for the optimal planning is based on a dynamic path inversion algorithm for which first the optimal velocity profile is determined and then the optimal steering signal is derived from a geometrical construction. A set of sufficient conditions for the feasibility of the velocity planning is devised and the practical computation of the optimal velocity profile uses time-discretization and linear programming. A worked example using $\eta^3$-splines illustrates the method.**

***Index Terms*— Automatic guided vehicles, time-optimal trajectory planning, dynamic path inversion, minimum-time control, feedforward control, constrained velocity planning.**

## I. INTRODUCTION

Nowadays, the handling of materials and parts through Automatic Guided Vehicles (AGVs) is of increasing importance in the automation and logistics of factories and warehouses. The absence of human intervention in the normal operations of the AGVs permits to optimize by design the performances and specifically to pursue a motion planning to achieve fastest movements with full respect of all the pertinent constraints [1], [2]. Considering the more general scenario of trajectory planning of wheeled mobile robots, the basic problem of minimum-time planning between two robot configurations has been addressed with (1) unobstructed environments and (2) obstructed environments with obstacle to be avoided (respectively cf. [3] and [4], [5] and references therein reported). The former case has been mainly dealt with the Pontryagin Maximum Principle (PMP) whereas with the latter, that is more difficult, a variety of sub-optimal or approximating techniques has been proposed e.g. potential functions, sampling methods such as Probabilistic Road Maps (PRMs), Rapidly-Exploring Dense Trees (RDTs), etc. Focusing on the special case of time-optimal (or minimum-time) trajectory planning on specific, desired paths, the use of the so-called "path-velocity decomposition" [6] permits to reduce the planning to a suitable optimal velocity problem. This was the approach pursued by Prado et al. [7] who presented a sub-optimal method based on path segmentation to achieve a smooth velocity planning suitable for both static and dynamic environments.

This article consider the problem of time-optimal trajectory planning of an AGV on a given feasible path while respecting velocity, acceleration and jerk constraints. Moreover, this planning must connect two arbitrary dynamic configurations of the AGV, i.e. at the start and at the end of the planning the AGV may not be at rest. A key to solve the problem is to recast it as a dynamic path inversion (DPI) problem. DPI, which was introduced in [8], is the problem, given a desired path on the output space, of finding the control inputs that generate the desired path. DPI is a variant of the more studied dynamic (signal) inversion which is the problem of finding the control inputs that generate the desired signal outputs [9]–[12].

The article is organized as follows. In Section II, Proposition 1 shows the connection for the AGV between the geometric continuity of its paths and the smoothness of its control inputs, and the formal definition of Time-Optimal dynamic Path Inversion (TOPI) problem is also presented. Section III is the core of the article. It exposes the devised dynamic path inversion algorithm that allows to apply the path-velocity decomposition concept so that the optimal velocity profile will be first independently determined and then the optimal steering signal will be found. The next Section IV rephrases the minimum-time constrained velocity planning problem into a state-space constrained minimum-time control problem. This permits to derive a set of sufficient conditions (Proposition 2) for which the velocity planning problem admits a solution. Subsequent Section V shows how to find an approximated determination of the optimal velocity profile by means of time-discretization and linear programming. An example of TOPI problem is presented and solved in Section VI. Brief conclusions are reported in Section VII.

## II. THE KINEMATIC MODEL AND THE TIME-OPTIMAL DYNAMIC PATH INVERSION PROBLEM

A typical wheeled automatic guided vehicle has forks for handling materials, two passive wheels and a motor wheel. See Figure 1 where a schematic plan view of an AGV and a Cartesian reference frame are depicted.

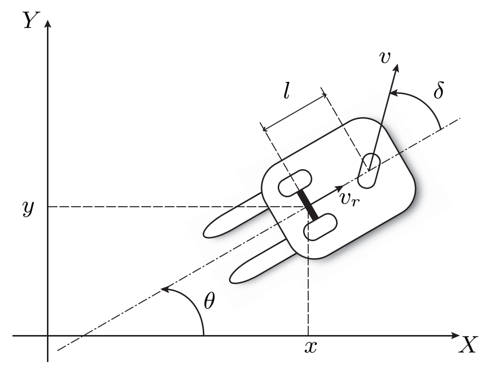


Fig. 1. A wheeled AGV on a Cartesian plane.

As usual, $x$ and $y$ indicate the Cartesian coordinates of the AGV rear-axle middle-point and $\theta$ is the vehicle orientation angle with respect of the $x$-axis. The motion of the AGV is actuated by the forward motor wheel whose linear velocity is $v$ and $\delta$ is the steering angle; $l$ is the distance between the rear-axle and the forward wheel's hub. With the usual modeling assumptions of no-slippage, rigid body and nonholonomic constraints the following nonlinear kinematic model of the AGV can be deduced [13]:

$$\begin{cases} \dot{x}(t) = v(t)\, \cos\theta(t)\, \cos\delta(t) \\ \dot{y}(t) = v(t)\, \sin\theta(t)\, \cos\delta(t) \\ \dot{\theta}(t) = \frac{1}{l}\, v(t)\, \sin\delta(t) \end{cases} \tag{1}$$

The linear velocity $v$ and the steering angle $\delta$ are the AGV control inputs.

The following definitions will be used along this paper.

*Definition 1:* A function $f : \mathbb{R} \to \mathbb{R}$, $t \to f(t)$ has a $PC^2$ continuity, and we write $f(t) \in PC^2$ if

1) $f(t) \in C^1(\mathbb{R})$ ,
2) $f(t) \in C^2(\mathbb{R} - \{t_1, t_2, \dots\})$ ,
3) $\exists \lim_{t\to t_i-} D^2 f(t)$ , $\exists \lim_{t\to t_i+} D^2 f(t)$ , $i = 1, 2, \dots$

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where $\{t_1, t_2, \dots\}$ is a set of possible discontinuity instants.

*Definition 2:* A Cartesian path $\Gamma$ has third order geometric continuity, and we say $\Gamma$ is a $G^3$-path, if its scalar curvature is continuous and the derivative with respect to the arc length of the curvature is continuous on the path too (for more details see [8]).

In order to obtain a smooth motion control, inputs $v$ and $\delta$ must be functions with $C^1$ continuity, i.e. they are continuous functions with continuous derivatives. A connections between smooth inputs and paths of the AGV is established by proposition below.

*Proposition 1:* Assign any $t_1 > 0$. If a Cartesian path $\Gamma$ is generated by the AGV with inputs $v(t), \delta(t) \in C^1([0, t_1])$ where $v(t) \neq 0$ and $|\delta(t)| < \frac{\pi}{2}$ $\forall t \in [0, t_1]$, then $\Gamma$ is a $G^3$-path. Conversely, given any $G^3$-path $\Gamma$ then exist inputs $v(t), \delta(t) \in C^1([0, t_1])$ with $v(t) \neq 0$ and $|\delta(t)| < \frac{\pi}{2}$ $\forall t \in [0, t_1]$, and initial conditions such that the path generated by the AGV coincides with the given $\Gamma$.

Proof of the above proposition can be deduced by a similar result proposed by Guarino Lo Bianco *et al.* in [8].

Instrumental to our approach to optimal motion control of AGVs is the definition of an "extended state" of system (1) that also comprises the control functions and their first derivatives:

$$\left\{x(t), y(t), \theta(t), v(t), \dot{v}(t), \delta(t), \dot{\delta}(t)\right\} .$$

Then, the following time-optimal dynamic path inversion (TOPI) problem can be posed.

*Definition 3:* **(TOPI problem)** Given an assigned $G^3$-path $\Gamma$, determine the control functions $v(t), \delta(t) \in PC^2$ such that system (1) travels exactly on path $\Gamma$ in minimum-time $t_f^*$ from initial extended state (at time $t = 0$)

$$\mathcal{A} \doteq \left\{x_A, y_A, \theta_A, v_A, \dot{v}_A, \delta_A, \dot{\delta}_A\right\} ,$$

to final extended state (at time $t = t_f^*$)

$$\mathcal{B} \doteq \left\{x_B, y_B, \theta_B, v_B, \dot{v}_B, \delta_B, \dot{\delta}_B\right\} ,$$

satisfying the following constraints

$$\begin{aligned} 0 \le v(t) \le v_M \quad & \forall t \in [0, t_f^*] , \\ |\dot{v}(t)| \le a_M \quad & \forall t \in [0, t_f^*] , \\ |\ddot{v}(t)| \le j_M \quad & \forall t \in [0, t_f^*] , \end{aligned}$$

where $v_M, a_M, j_M > 0$ are given bounds.

Hence, in order to give a solution for TOPI problem, it is preliminarily necessary to determine a desired $G^3$-path that satisfies the interpolating data deduced from the extended states $\mathcal{A}$ and $\mathcal{B}$ [14]. Let us introduce the following relations

$$v_r(t) = v(t)\,\cos\delta(t) , \tag{2}$$

$$\dot{v}_r(t) = \dot{v}(t)\,\cos\delta(t) - v(t)\,\dot{\delta}(t)\,\sin\delta(t) , \tag{3}$$

$$\omega(t) = \frac{1}{l}\,v(t)\,\sin\delta(t) , \tag{4}$$

$$\dot{\omega}(t) = \frac{1}{l}\,\dot{v}(t)\,\sin\delta(t) + \frac{1}{l}\,v(t)\,\dot{\delta}(t)\,\cos\delta(t) , \tag{5}$$

where $v_r(t)$ and $\dot{v}_r(t)$ denote the linear velocity and acceleration of the AGV rear-axle middle-point, and $\omega(t)$ and $\dot{\omega}(t)$ denote the angular velocity and acceleration of the AGV. From [8], the curvature and its derivative with respect to the arclength, $k_A$ and $\dot{k}_A$ in $t = 0$, and $k_B$ and $\dot{k}_B$ in $t = t_f$, are given by

$$k_A = \frac{\omega_A}{v_{rA}} , \; \dot{k}_A = \frac{\dot{\omega}_A\, v_{rA} - \omega_A\, \dot{v}_{rA}}{v_{rA}^3} , \tag{6}$$

and

$$k_B = \frac{\omega_B}{v_{rB}} , \; \dot{k}_B = \frac{\dot{\omega}_B\, v_{rB} - \omega_B\, \dot{v}_{rB}}{v_{rB}^3} , \tag{7}$$

where $\omega_A = \omega(0)$, $v_{rA} = v_r(0)$, $\omega_B = \omega(t_f^*)$ and $v_{rB} = v_r(t_f^*)$. By substituting relations (2)-(5) in (6)-(7), the following equations are obtained

$$k_A = \frac{1}{l}\,\tan\delta_A , \tag{8}$$

$$\dot{k}_A = \frac{1}{l}\,\frac{\dot{\delta}_A}{v_A\,\cos^3\delta_A} , \tag{9}$$

and

$$k_B = \frac{1}{l}\,\tan\delta_B , \tag{10}$$

$$\dot{k}_B = \frac{1}{l}\,\frac{\dot{\delta}_B}{v_B\,\cos^3\delta_B} . \tag{11}$$

On the extended states $\mathcal{A}$ and $\mathcal{B}$ we impose the assumptions

$$|\delta_A| < \pi/2 \;\text{ and }\; |\delta_B| < \pi/2 .$$

Therefore, relations (9) and (11) indicate that there exist two definite forbidden cases

$$\{v_A = 0\} \wedge \{\dot{\delta}_A \neq 0\} , \; \{v_B = 0\} \wedge \{\dot{\delta}_B \neq 0\} ,$$

which are considered as further assumptions on the TOPI problem. On the other hand if $v_A = 0$ and $\dot{\delta}_A = 0$ and similarly $v_B = 0$ $\dot{\delta}_B = 0$, then $\dot{k}_A$ and $\dot{k}_B$ can be arbitrarily assigned and this improves the design freedom in shaping the $\Gamma$ path for the AGV.

Hence, the $G^3$-path $\Gamma$ must satisfy at the endpoints the interpolations conditions shown in Figure 2, i.e. the initial and final Cartesian points of $\Gamma$ have $(x_A, y_A)$ and $(x_B, y_B)$ as coordinates, $\theta_A$ and $\theta_B$ as unit-tangent directions, $k_A$ and $k_B$ as curvatures, $\dot{k}_A$ and $\dot{k}_B$ as curvature derivatives respectively.

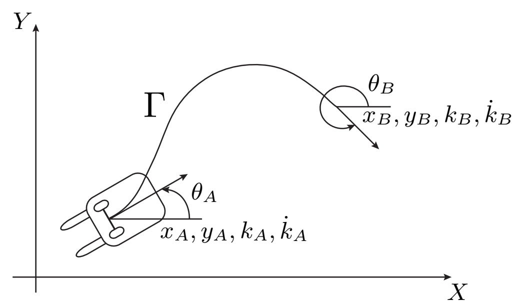


Fig. 2. The interpolations conditions at the endpoints of path $\Gamma$.

This interpolation problem can be easily solved using the $\boldsymbol{\eta}^3$-splines [14], [15] which are seventh-order polynomial curves with free design parameters (the $\boldsymbol{\eta}$ vector) to shape the desired path intercourse between the endpoints.

*Remark 1:* In this article, path $\Gamma$ denotes the Cartesian path generated by the rear-axle middle-point, i.e. by $(x(t), y(t))$. The next section will introduce the other relevant path of the AGV, denoted by $\Gamma_f$, that is the path generated by its forward motor wheel.

## III. The Dynamic Path Inversion Algorithm

The time-optimal control functions $v^*(t)$ and $\delta^*(t)$, which permit the AGV to follow the given path $\Gamma$ in minimum-time, will be obtained by a dynamic path inversion procedure.

Note that functions $v^*(t)$ and $\delta^*(t)$, solution of the TOPI problem, are associated to the actuated motor wheel of the AGV (see Figure 1), so that the inversion procedure will need to determine the path $\Gamma_f$ of the forward wheel which is geometrically linked to $\Gamma$. Knowledge of $\Gamma_f$ and its total distance $s_f$ allows to apply the path-velocity decomposition method [6] to the TOPI problem so that the velocity $v^*(t)$ will be computed independently from

$\delta^*(t)$ by setting a minimum-time constrained velocity planning. Then the optimal steering $\delta^*(t)$ will be determined by exploiting the geometric properties of model (1) relative to paths $\Gamma$ and $\Gamma_f$.

The dynamic path inversion algorithm can be then described in the following three steps:

1) Determine the path $\Gamma_f$ of the forward wheel. Consider the following parametrization of path $\Gamma$ (as customary using $\boldsymbol{\eta}^3$-splines)

$$\mathbf{p}(u): \quad \begin{array}{l} [0,1] \to \mathbb{R}^2 \\ u \to \mathbf{p}(u) \, . \end{array}$$

The unit tangent vector $\boldsymbol{\tau}(u)$ of $\Gamma$ is given by

$$\boldsymbol{\tau}(u) = \frac{\dot{\mathbf{p}}(u)}{\|\dot{\mathbf{p}}(u)\|} \, ,$$

and a parametrization of path $\Gamma_f$ can be obtained as follows

$$\mathbf{p}_f(u) = \mathbf{p}(u) + l\, \boldsymbol{\tau}(u), \; u \in [0,1], \tag{12}$$

where $l$ is the distance between the rear-axle middle point and the forward wheel. Figure 3 depicts the geometric relation between paths $\Gamma$ and $\Gamma_f$.

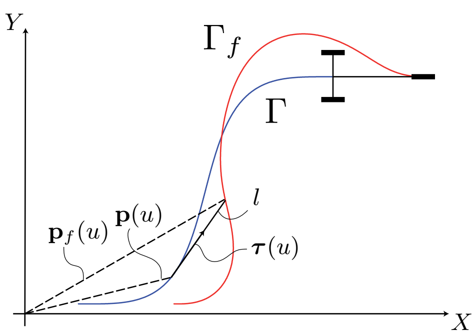


Fig. 3. Geometric construction of the forward path $\Gamma_f$.

Compute the total distance $s_f$ to be travelled by the forward wheel on $\Gamma_f$:

$$s_f = \int_0^1 \|\dot{\mathbf{p}}_f(u)\| du \, . \tag{13}$$

2) Determine the minimum-time velocity $v^*(t)$ by solving the following constrained problem:

$$\min_{v \in PC^2} t_f \tag{14}$$

such that

$$\int_0^{t_f} v(\xi) d\xi = s_f \tag{15}$$

$$v(0) = v_A \, , \quad v(t_f) = v_B \tag{16}$$

$$\dot{v}(0) = a_A \, , \quad \dot{v}(t_f) = a_B \tag{17}$$

$$0 \le v(t) \le v_M \quad \forall t \in [0, t_f] \, , \tag{18}$$

$$|\dot{v}(t)| \le a_M \quad \forall t \in [0, t_f] \, , \tag{19}$$

$$|\ddot{v}(t)| \le j_M \quad \forall t \in [0, t_f] \, , \tag{20}$$

where $a_A := \dot{v}_A$, $a_B := \dot{v}_B$ are the linear accelerations of the forward wheel at states $\mathcal{A}$ and $\mathcal{B}$ respectively (cf. definition (3) of the TOPI problem). The solution of the above problem is $v^*(t) \in PC^2$ with associated minimum-time $t_f^*$.

3) Determine the optimal steering function $\delta^*(t)$ by solving the following equation system:

$$\left\{ \begin{array}{l} \int_0^t v^*(\xi) d\xi = \int_0^u \|\dot{\mathbf{p}}_f(\xi)\| d\xi \\ \delta^*(t) = \arg \boldsymbol{\tau}_f(u) - \arg \boldsymbol{\tau}(u) \end{array} \right. . \tag{21}$$

The geometrical meaning of this determination is depicted in Figure 4.

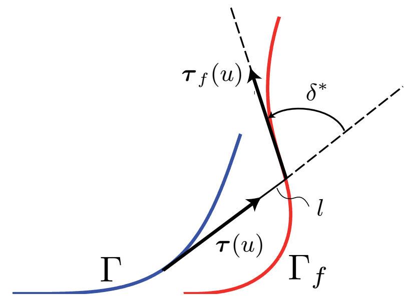


Fig. 4. Geometrical interpretation of equation system (21).

*Remark 2:* The velocity planning problem (14)-(20) leads to a smooth velocity profile that is easy to implement on an actuator drive because velocity and acceleration are continuous and the jerk (the time-derivative of acceleration) is limited and constrained as desired (by setting the bound $j_M$). Also note in (18) that the constraint $v(t) \ge 0$ imposes that the automatic guided vehicle does not go backward on the desired path.

## IV. The Minimum-Time Constrained Velocity Planning and a Sufficient Condition

The minimum-time constrained velocity planning problem defined in (14)-(20) can be easily recast into a minimum-time control problem with respect to a suitable state-space system. Indeed consider the jerk $\ddot{v}(t)$ as the control input $u(t)$ of the cascade of three integrators as depicted in Figure 5. Introduce the state $\mathbf{x}(t)$

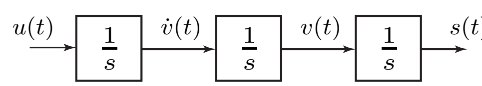


Fig. 5. The system model for velocity planning.

as the following column vector

$$\mathbf{x}(t) := \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} := \begin{bmatrix} s(t) \\ v(t) \\ \dot{v}(t) \end{bmatrix} .$$

Then, the system equations are given by

$$\dot{\mathbf{x}}(t) = \mathbf{A}\, \mathbf{x}(t) + \mathbf{B}\, u(t) \, , \tag{22}$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} .$$

Constraints (18) and (19) can be considered as state constraints and (20) is an input constraint of system (22). Hence, problem (14)-(20) is equivalent to finding a time-optimal (minimum-time) open-loop control $u^*(t)$ that brings system (22) from the initial state $\mathbf{x}(0) = [0 \; v_A \; a_A]^T$ to the final state $\mathbf{x}(t_f^*) = [s_f \; v_B \; a_B]^T$ in minimum-time $t_f^*$, while satisfying the following constraints

$$0 \le x_2(t) \le v_M \quad \forall t \in [0, t_f^*] \, , \tag{23}$$

$$|x_3(t)| \le a_M \quad \forall t \in [0, t_f^*] \,, \tag{24}$$

and

$$|u^*(t)| \le j_M \quad \forall t \in [0, t_f^*] \,. \tag{25}$$

A sufficient condition that guarantees the solvability of problem (14)-(20) is given by the following proposition.

*Proposition 2:* The minimum-time constrained velocity planning problem (14)-(20) has solution if the following conditions are satisfied:

$$0 \le v_A \le v_M \,, \qquad 0 \le v_B \le v_M \,, \tag{26}$$

$$|a_A| \le a_M \,, \qquad |a_B| \le a_M \,, \tag{27}$$

$$\text{if } a_A \ge 0 \text{ then} \quad v_A + \frac{1}{2}\,\frac{a_A^2}{j_M} \le v_M \,, \tag{28}$$

$$\text{if } a_A < 0 \text{ then} \quad v_A - \frac{1}{2}\,\frac{a_A^2}{j_M} \ge 0 \,, \tag{29}$$

$$\text{if } a_B \ge 0 \text{ then} \quad v_B - \frac{1}{2}\,\frac{a_B^2}{j_M} \ge 0 \,, \tag{30}$$

$$\text{if } a_B < 0 \text{ then} \quad v_B + \frac{1}{2}\,\frac{a_B^2}{j_M} \le v_M \,, \tag{31}$$

and

$$s_f \ge s_{ref} \,, \tag{32}$$

where $s_{ref}$ is a reference distance depending on the problem data which is defined below by a four-step procedure:

1)
$$s_1 := \frac{v_A\,|a_A|}{j_M} + \frac{1}{3}\,\frac{a_A^3}{j_M^2} \quad \text{and} \quad v_1 := v_A + \operatorname{sgn}(a_A)\,\frac{1}{2}\,\frac{a_A^2}{j_M} \,.$$

2)
$$s_2 := \frac{v_B\,|a_B|}{j_M} - \frac{1}{3}\,\frac{a_B^3}{j_M^2} \quad \text{and} \quad v_2 := v_B - \operatorname{sgn}(a_B)\,\frac{1}{2}\,\frac{a_B^2}{j_M} \,.$$

3) if $\sqrt{j_M\,|v_1 - v_2|} \le a_M$ then

$$v_{ref} := \max(v_1, v_2) \,,$$

$$s_c := \frac{2\,v_{ref}\,\sqrt{j_M\,|v_1 - v_2|}}{j_M} - \frac{[j_M\,|v_1 - v_2|]^{3/2}}{j_M^2} \,,$$

else

$$s_c := \frac{1}{2}\,\frac{|v_1^2 - v_2^2|}{a_M} + \frac{1}{2}\,\frac{a_M\,(v_1 + v_2)}{j_M} \,.$$

4) $s_{ref} := s_1 + s_c + s_2$ .

*Proof:* For brevity the proof is omitted.

## V. Approximating the Minimum-Time Velocity Using Linear Programming

This section presents a numerically approximated solution to problem (23)-(25). The technique is based on the discretization of system (22) and on finding by means of linear programming a time-optimal discrete-time control sequence. This idea draws on the generalized bang-bang control concept recently exposed in [16].

An approximation to the minimum-time velocity profile can be found by procedure below:

1) Choose a sampling period $T$ and find the discretized system of (22).
2) Find the time-optimal input sequence $u_T^*(k)$.
3) The time-optimal continuous-time jerk $u^*(t)$ can be obtained from $u_T^*(k)$ with a zero-order hold

$$u^*(t) = u_T^*\left(\lfloor \frac{t}{T} \rfloor\right) \,,$$

where $\lfloor x \rfloor = \max\{z \in \mathbb{Z} \,:\, z \le x\}$ denotes the integer part of $x$.
4) Determine the minimum-time velocity planning $v^*(t)$ by double integration of $u^*(t)$.

Note that if $T \to 0$, then the approximated solution converges to the optimal continuous-time one (cf. [16]).

The core of the procedure is evidently Step 2 where the minimum-time control sequence $u_T^*(k)$, $k = 0, 1, \ldots, k_f^* - 1$, can be found trough linear programming because constraints (23)-(25) can be represented as linear inequalities. Indeed, $u_T^*(k)$ and the associated minimum number of steps $k_f^*$ needed for the requested constrained transition can be found by solving a sequence of linear programming feasibility tests.

The matrices of the equivalent discrete-time system of (22) are

$$\mathbf{A}_d = e^{\mathbf{A}\,T} = \begin{bmatrix} 1 & T & \frac{1}{2}T^2 \\ 0 & 1 & T \\ 0 & 0 & 1 \end{bmatrix} \,,$$

and

$$\mathbf{B}_d = \left(\int_0^T e^{\mathbf{A}\,\tau} d\tau\right) \mathbf{B} = \begin{bmatrix} \frac{1}{6}T^3 \\ \frac{1}{2}T^2 \\ T \end{bmatrix} \,,$$

where $T$ is the sampling period. Then, the discrete-time system is

$$\mathbf{x}(k+1) = \mathbf{A}_d\,\mathbf{x}(k) + \mathbf{B}_d\,u(k) \,,$$

whose solution is given by

$$\mathbf{x}(k) = \mathbf{A}_d^k\,\mathbf{x}_0 + \sum_{j=0}^{k-1} \mathbf{A}_d^{k-1-j}\mathbf{B}_d\,u(j) \,, \tag{33}$$

where

$$\mathbf{x}(k) = \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} \text{ and } x_0 := \mathbf{x}(0) = \begin{bmatrix} 0 \\ v_A \\ a_A \end{bmatrix} \,.$$

Denote with $k_f \in \mathbb{N}$ the input sequence length and the control vector $\mathbf{u} \in \mathbb{R}^{k_f}$ as follows

$$\mathbf{u} = \begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(k_f - 1) \end{bmatrix} \,.$$

From (25), it must be

$$-j_M \cdot \mathbf{1}_{k_f} \le \mathbf{u} \le j_M \cdot \mathbf{1}_{k_f}$$

where $\mathbf{1}_{k_f}$ denotes the $k_f$-dimensional vector whose components are all equal to 1. The velocity constraint for discrete-time system is given by

$$0 \le x_2(k) \le v_M \,, \quad \text{with} \quad k = 0, \ldots, k_f - 1 \,. \tag{34}$$

From equation (33), velocity sequence $x_2(k)$ can be written as follows

$$\begin{aligned} x_2(k) &= \mathbf{C}_1\,\mathbf{x}(k) \\ &= \mathbf{C}_1\left[\mathbf{A}_d^k\,\mathbf{x}_0 + \sum_{j=0}^{k-1} \mathbf{A}_d^{k-1-j}\mathbf{B}_d\,u(j)\right] \\ &= \mathbf{C}_1\,\mathbf{A}_d^k\,\mathbf{x}_0 + \sum_{j=0}^{k-1} \mathbf{C}_1\mathbf{A}_d^{k-1-j}\mathbf{B}_d\,u(j) \,, \end{aligned} \tag{35}$$

where

$$\mathbf{C}_1 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \,.$$

By substituting (35) in (34), the following inequalities are obtained

$$-\mathbf{C}_1\,\mathbf{A}_d^k\,\mathbf{x}_0 \le \sum_{j=0}^{k-1} \mathbf{C}_1 \mathbf{A}_d^{k-1-j} \mathbf{B}_d\,\mathbf{u}(j) \le v_M - \mathbf{C}_1\,\mathbf{A}_d^k\,\mathbf{x}_0\,,$$

with $k = 0, \ldots, k_f - 1$. Set $\mathbf{v}_c = v_M \cdot \mathbf{1}_f$, then these above inequalities can be concisely rewritten as follows

$$-\mathbf{G}_1 \le \mathbf{H}_1\,\mathbf{u} \le \mathbf{v}_c - \mathbf{G}_1\,,$$

where $\mathbf{G}_1 \in \mathbb{R}^{k_f}$ and $\mathbf{H}_1 \in \mathbb{R}^{k_f \times k_f}$ are given by

$$\mathbf{G}_1 = \begin{bmatrix} \mathbf{C}_1\,\mathbf{x}_0 \\ \mathbf{C}_1\,\mathbf{A}_d\,\mathbf{x}_0 \\ \mathbf{C}_1\,\mathbf{A}_d^2\,\mathbf{x}_0 \\ \vdots \\ \mathbf{C}_1\,\mathbf{A}_d^{k_f-1}\,\mathbf{x}_0 \end{bmatrix},$$

and

$$\mathbf{H}_1 = \begin{bmatrix} \mathbf{C}_1\,\mathbf{B}_d & \mathrm{O} & \cdots & \mathrm{O} & \mathrm{O} \\ \mathbf{C}_1\,\mathbf{A}_d\,\mathbf{B}_d & \mathbf{C}_1\,\mathbf{B}_d & \ddots & & \mathrm{O} \\ \mathbf{C}_1\,\mathbf{A}_d^2\,\mathbf{B}_d & \mathbf{C}_1\,\mathbf{A}_d\,\mathbf{B}_d & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \mathrm{O} \\ \mathbf{C}_1\,\mathbf{A}_d^{k_f-1}\,\mathbf{B}_d & \cdots & \cdots & \cdots & \mathbf{C}_1\,\mathbf{B}_d \end{bmatrix}.$$

The acceleration constraint for the discrete-time system is given by

$$-a_M \le x_3(k) \le a_M\,, \quad \text{with} \quad k = 0, \ldots, k_f - 1\,. \tag{36}$$

Set $\mathbf{a}_c = a_M \cdot \mathbf{1}_f$ and

$$\mathbf{C}_2 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix},$$

then constraints (36) can be written as

$$-\mathbf{a}_c - \mathbf{G}_2 \le \mathbf{H}_2\,\mathbf{u} \le \mathbf{a}_c - \mathbf{G}_2\,,$$

where $\mathbf{G}_2 \in \mathbb{R}^{k_f}$ and $\mathbf{H}_2 \in \mathbb{R}^{k_f \times k_f}$ are given by

$$\mathbf{G}_2 = \begin{bmatrix} \mathbf{C}_2\,\mathbf{x}_0 \\ \mathbf{C}_2\,\mathbf{A}_d\,\mathbf{x}_0 \\ \mathbf{C}_2\,\mathbf{A}_d^2\,\mathbf{x}_0 \\ \vdots \\ \mathbf{C}_2\,\mathbf{A}_d^{k_f-1}\,\mathbf{x}_0 \end{bmatrix},$$

and

$$\mathbf{H}_2 = \begin{bmatrix} \mathbf{C}_2\,\mathbf{B}_d & \mathrm{O} & \cdots & \mathrm{O} \\ \mathbf{C}_2\,\mathbf{A}_d\,\mathbf{B}_d & \ddots & \ddots & \mathrm{O} \\ \mathbf{C}_2\,\mathbf{A}_d^2\,\mathbf{B}_d & \ddots & \ddots & \mathrm{O} \\ \vdots & \ddots & \ddots & \vdots \\ \mathbf{C}_2\,\mathbf{A}_d^{k_f-1}\,\mathbf{B}_d & \cdots & \cdots & \mathbf{C}_2\,\mathbf{B}_d \end{bmatrix}.$$

The interpolation condition on the final state is

$$\mathbf{x}_f := \mathbf{x}(k_f) = \begin{bmatrix} s_f \\ v_f \\ a_f \end{bmatrix}. \tag{37}$$

From equation (33) we obtain the final state interpolation condition as follows

$$\mathbf{H}_{eq}\,\mathbf{u} = \mathbf{x}_f - \mathbf{A}_d^{k_f}\,\mathbf{x}_0\,,$$

where $\mathbf{H}_{eq} \in \mathbb{R}^{3 \times k_f}$ is given by

$$\mathbf{H}_{eq} = \begin{bmatrix} \mathbf{A}_d^{k_f-1}\,\mathbf{B}_d & \mathbf{A}_d^{k_f-2}\,\mathbf{B}_d & \cdots & \mathbf{B}_d \end{bmatrix}.$$

In conclusion given a number of steps $k_f$, there exists an input vector $\mathbf{u}$ for which the constraints on velocity, acceleration and jerk, and the final interpolation condition are satisfied if and only if the following linear programming problem is feasible, i.e. admits a solution

$$\begin{cases} -u_M \cdot \mathbf{1}_{k_f} \le \mathbf{u} \le u_M \cdot \mathbf{1}_{k_f} \\ -\mathbf{G}_1 \le \mathbf{H}_1\,\mathbf{u} \le \mathbf{v}_c - \mathbf{G}_1 \\ -\mathbf{a}_c - \mathbf{G}_2 \le \mathbf{H}_2\,\mathbf{u} \le \mathbf{a}_c - \mathbf{G}_2 \\ \mathbf{H}_{eq}\,\mathbf{u} = \mathbf{x}_f - \mathbf{A}_d^{k_f}\,\mathbf{x}_0\,. \end{cases} \tag{38}$$

The minimum number of steps $k_f^*$ and the associated optimal discrete-time control sequence $u_T^*(k)$, with $k = 0, \ldots, k_f^* - 1$, can be found by solving a sequence of linear programming feasibility tests, defined by (38), through a simple bisection algorithm. For details on the linear programming approach to minimum-time control refer to [16].

## VI. An example

Consider an AGV with $l = 1.1$ [m], the distance between the motor wheel and the rear-axle, and constraints on the actuation of the motor wheel given by $v_M = 3$ [m/s], $a_M = 1$ [m/s$^2$], $j_M = 0.5$ [m/s$^3$].

It is desired a minimum-time transition between the extended states $\mathcal{A}$ and $\mathcal{B}$ given by (measures are expressed in m, m/s, m/s$^2$, rad, rad/s):

$$\mathcal{A} = \left\{ x_A, y_A, \theta_A, v_A, a_A, \delta_A, \dot{\delta}_A \right\} = \{0, 0, 0, 1, -1, 0, 0\}\ .$$

$$\mathcal{B} = \left\{ x_B, y_B, \theta_B, v_B, a_B, \delta_B, \dot{\delta}_B \right\} = \{16, 8, 0, 3, 0, 0, 0\}\ .$$

The desired Cartesian path $\Gamma$ between $(x_A, y_A)$ and $(x_B, y_B)$ is an S-shaped path that can be easily determined by interpolation with the $\boldsymbol{\eta}^3$-splines [14]. The interpolation data is $(\theta_A, k_A, \dot{k}_A)$ and $(\theta_B, k_B, \dot{k}_B)$ for which $\theta_A = 0$ and $\theta_B = 0$ from the assigned extended states $\mathcal{A}$ and $\mathcal{B}$ and $k_A = 0$, $\dot{k}_A = 0$ and $k_B = 0$, $\dot{k}_B = 0$ as it follows from relations (8)-(11).

Path $\Gamma$ is then an $\boldsymbol{\eta}^3$-spline, a seventh order polynomial curve, whose free parameters are chosen according to the heuristic rule suggested in [14], [17]:

$$\boldsymbol{\eta} = (\eta_1, \eta_2, \eta_3, \eta_4, \eta_5, \eta_6) = (d, d, 0, 0, 0, 0)$$

where $d = \|(x_A - x_B, y_A - y_B)\| = 17.89$ is the Euclidean distance between $(x_A, y_A)$ and $(x_B, y_B)$. Path $\Gamma$ is the blue one depicted in Figure 6.

To determine the time-optimal controls $v^*(t)$ and $\delta^*(t)$ which are the solution to the TOPI problem we use the dynamic path inversion algorithm described in three steps in Section III.

**Step 1:** The path $\Gamma_f$ of the forward motor wheel is computed according to (12). It is depicted in Figure 6. The length of $\Gamma_f$ is $s_f = 19.12$ [m] according to (13).

**Step 2:** The existence of the time-optimal velocity is guaranteed by the fulfilment of the sufficient conditions of Proposition 2. Indeed, conditions (26) and (27) are immediately satisfied. Because $a_A < 0$ and $a_B \ge 0$, we check conditions (29) and (30) respectively: $v_A + \frac{1}{2}\frac{a_A^2}{j_M} = 0 \ge 0$ and $v_B - \frac{1}{2}\frac{a_B^2}{j_M} = 3 \ge 0$. Application of the four-step procedure of Proposition 2 determines $s_{ref} = 8.17$ [m] so that the last inequality (32) is also satisfied: $s_f \ge s_{ref}$. Hence, since the constrained minimum-time velocity problem (14)-(20) has solution, the TOPI problem has solution too by virtue of the path dynamic inversion algorithm of Section III.

The approximated determination of $v^*(t)$ is gained with the procedure detailed in Section V and its profile is shown in Figure 7. It has been chosen the sampling time $T = 0.01$ [s] and the linear programming routine has run using MOSEK [18]. The resulting minimum-time for the transition of the AGV from $\mathcal{A}$ to $\mathcal{B}$ along $\Gamma$ is $t_f^* = 10.64$ [s].

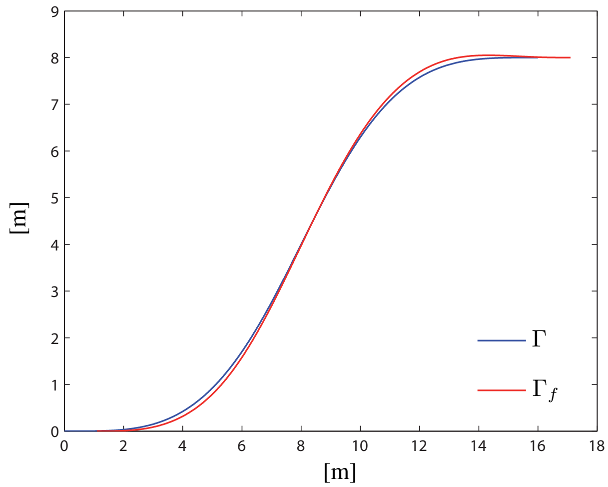


Fig. 6. The planned path $\Gamma$ and the associated forward path $\Gamma_f$ of the AGV.

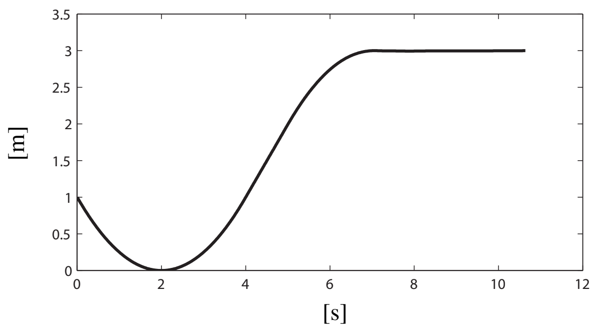


Fig. 7. The optimal velocity profile $v^*(t)$.

**Step 3:** The optimal steering control $\delta^*(t)$ is determined by solving (21) with a sweeping discretization on parameter $u \in [0, 1]$. The result is shown in Figure 8. Many simulations of the motion of the AGV have been performed. In particular, the minimum-time transition of the AGV between the extended states $\mathcal{A}$ and $\mathcal{B}$ along $\Gamma$ has been simulated by using the found $v^*(t)$ and $\delta^*(t)$. The adopted approximations are good enough to ensure a tracking of the planned trajectory with negligible errors.

## VII. Conclusions

The article has presented a feedforward technique for minimum-time motion control of an automatic guided vehicle on a given $G^3$-path. Implementing this technique requires a suitable feedback action on the AGV to compensate or correct the trajectory errors due to mis-modeling, neglected frictions, etc. This can be achieved by using various means. An effective one may be the convex replanning approach exposed in [19].

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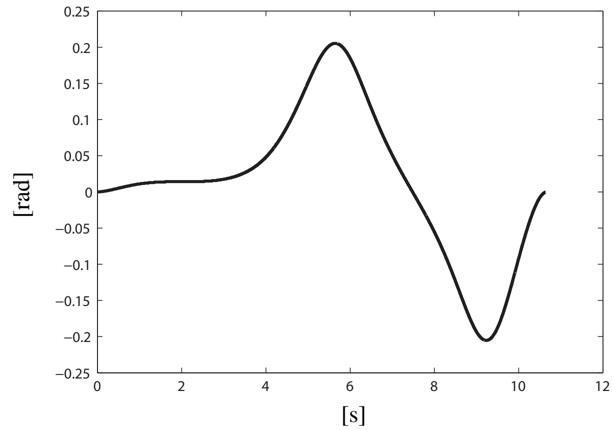


Fig. 8. The optimal steering control $\delta^*(t)$.

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