Feedforward/feedback Control of a Magnetic Levitation Apparatus

Antonio Di Fluri, Aurelio Piazzi and Antonio Visioli

Abstract— **A magnetic levitation apparatus is often employed in control education as a benchmark for the design of a control system for a nonlinear unstable process. In this paper we present a combined feedforward/feedback approach which takes into account the constraints of the system. In particular, we highlight the use of a feedforward control law which is based on a dynamic inversion of the closed-loop system. The robustness of the control system with respect to structured uncertainties is evaluated.**

I. INTRODUCTION

In control education it is essential to make students aware of problems that need to be tackled when facing the design of control systems for real processes, for example the presence of constraints, uncertainties, and so on. From another point of view it is useful to provide challenging benchmark problems where the design of the control technique that meets the required specifications is not trivial. In this context, it has to be stressed that the main purpose of feedback control is to reduce the effect of modelling uncertainties and to compensate for disturbances, while the use of a suitably design open-loop control action can be exploited to increase the set-point following performance. Indeed, the combined synthesis of a feedforward/feedback control can play a key role in achieving a high performance and in meeting tight control requirements.

A magnetic levitation apparatus is often employed in control education as a benchmark for the design of a control system for a nonlinear unstable process. In this paper we propose the use of a state-feedback control law combined with a feedforward control law based on the dynamic inversion of the obtained closed-loop system [1], [2], [3]. In particular, two feedback control schemes are considered and we show that the use of the so-called transition polynomials [4] as desired output functions allows to take into account the system constraints. The robustness of the control system is eventually evaluated and this allows to determine which of the control schemes gives the best performance.

The paper is organised as follows. The model of the magnetic levitation apparatus is presented in Section 2. The design of the state-feedback control law is addressed in Section 3, while that of the dynamic-inversion-based feedforward control law is explained in Section 4. Simulation results are shown in Section 5 while conclusions are drawn in Section 6.

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II. MODELLING

A sketch of the considered magnetic levitation apparatus is shown in Figure 1 where v and i are, respectively, the input voltage and the current applied to the electromagnet, R is the resistance of the wire, m is the mass of the object and y is its position. From the second dynamic law, we have that the system can be described by the following differential equation:

$$m\frac{d^2}{dt^2}y(t) = -f_b(t) + mg + F(t) \tag{1}$$

where f_b is the friction force exerted by the air, g is the gravity acceleration and F is the magnetic force. The energy of the electromagnet can be expressed as

$$E(t, y(t)) = \frac{1}{2}L(y(t))i(t)^2 \tag{2}$$

where the inductance L of the electromagnet depends on the position of the mass according to the expression

$$L(y(t)) = \frac{\lambda}{1 + \mu y(t)} \tag{3}$$

where λ is the value of the inductance when the mass is in contact with the electromagnet and μ is a parameter that describes the influence of the mass position on the inductance. We have therefore that the magnetic force can be expressed as

$$F(i(t), y(t)) = \frac{\partial E(t, y(t))}{\partial y} = -\frac{1}{2}\frac{\lambda\mu i(t)^2}{(1 + \mu y(t))^2}. \tag{4}$$

By expressing the friction force exerted by the air as

$$f_b(t) = b\frac{d}{dt}y(t) \tag{5}$$

where b is the viscous friction coefficient, and by taking into account (4), expression (1) can be rewritten as

$$m\frac{d^2}{dt^2}y(t) = -b\frac{d}{dt}y(t) + mg - \frac{1}{2}\frac{\lambda\mu i(t)^2}{(1 + \mu y(t))^2}. \tag{6}$$

Then, by considering that

$$v(t) = Ri(t) + \frac{d}{dt}(Li(t)) \tag{7}$$

and by taking into account (3) we have that the input voltage can be expressed as

$$v(t) = Ri(t) + -\frac{\lambda\mu i(t)}{(1 + \mu i(t))^2} + \frac{\lambda}{1 + \mu y(t)}\frac{d}{dt}i(t). \tag{8}$$

Finally, by defining the state variables as $x_1 = y$ (mass position), $x_2 = \dot{y}$ (mass velocity) and $x_3 = i$ (current applied

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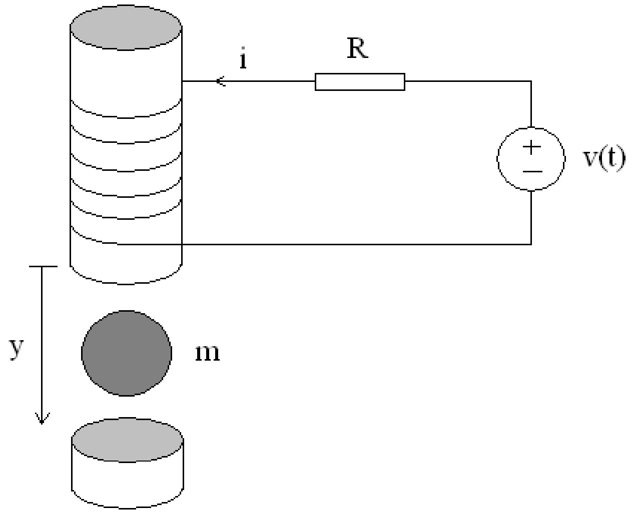

Fig. 1. Sketch of the magnetic levitation apparatus.

TABLE I
VALUE OF THE SYSTEM PARAMETERS.

b [Ns/m]	λ [H]	m [kg]	μ [1/m]	R [Ω]	g [m/s^2]
$1.82 \cdot 10^{-5}$	0.64	0.001	2	47	9.8

to the electromagnet), we obtain the state-space model of the system:

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= g - \frac{b}{m}x_2(t) - \frac{\lambda\mu x_3(t)^2}{2m(1+\mu x_1(t))^2} \\ \dot{x}_3(t) &= \frac{1+\mu x_1(t)}{\lambda}\left[-Rx_3(t) + \frac{\lambda\mu}{(1+\mu x_1(t))^2}x_2(t)x_3(t) + v(t)\right] \\ y(t) &= x_1(t) \end{aligned} \tag{9}$$

The values of the parameters considered in this paper are shown in Table I.

III. PROBLEM FORMULATION AND STATE-FEEDBACK CONTROL DESIGN

The control problem to be addressed is to move the mass from an initial position y_0 to a final position y_1 (without loss of generality we consider hereafter $y_1 > y_0$) by minimising the settling time subject to the constraints of the system, which are:

- the minimum and maximum input voltage are -10 V and +10 V respectively;
- the current to be applied to the electromagnet must be positive because the electromagnet is capable only to attract the object regardless of the current sign. Moreover, changing the sign of the current might cause problems to the electromagnet because of the hysteresis phenomenon;
- since the electromagnet is capable only to attract the object, the maximum positive acceleration of the object is the gravity acceleration (for simplicity, the friction force exerted by the air is neglected in this context).

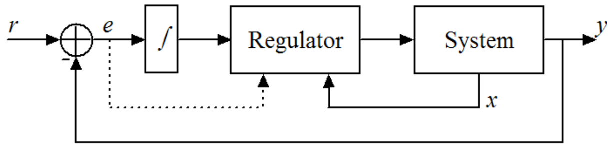

Fig. 2. The control scheme.

TABLE II
VALUE OF THE PARAMETERS OF THE STATE-SPACE FEEDBACK CONTROLLER.

	k_{11}	k_{12}	k_{13}	k_2	k_3
$k_3 = 0$	-20	-1.9	-30	58	0
$k_3 \neq 0$	-24	-9.5	-4	148	94

A state-feedback controller can be designed to meet the control specifications. Actually, being a set-point following task, an integral action has to be added to achieve robust regulation. This can be done in two different ways, as shown in the scheme of Figure 2 where the dotted line is optional. Indeed, the control variable is determined by a combination of the three states of the system, of the integral of the control error (namely, the difference between the command reference signal r and the system output y) and (optionally) of the control error itself. Formally, the control variable $v(t)$ is determined as

$$v(t) = -\begin{bmatrix} k_{11} & k_{12} & k_{13} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} - k_2 x_4(t) - k_3 e(t) \tag{10}$$

where $\dot{x}_4(t) = e(t) = r(t) - y(t)$ and k_3 can be zero. The state-space equations of the feedback control systems can be then written as:

$$\begin{aligned} \dot{x}_1(t) =&\ x_2(t) \\ \dot{x}_2(t) =&\ g - \frac{b}{m}x_2(t) - \frac{\lambda\mu x_3(t)^2}{2m(1+\mu x_1(t))^2} \\ \dot{x}_3(t) =&\ \frac{1+\mu x_1(t)}{\lambda}\Big[-Rx_3(t) + \frac{\lambda\mu}{(1+\mu x_1(t))^2}x_2(t)x_3(t) \\ &-k_{11}x_1(t) - k_{12}x_2(t) - k_{13}x_3(t) - k_2x_4(t) \\ &-k_3(r(t) - x_1(t)\Big] \\ \dot{x}_4(t) =&\ r(t) - x_1(t) \\ y(t) =&\ x_1(t) \end{aligned} \tag{11}$$

The values of the controller parameters k_{11}, k_{12}, k_{13}, k_2, k_3 can be found, with the help of a linearised model of the system and of the pole assignment theorem [5] (and of a simulation tool), by applying a trial-and-error procedure in order to obtain an acceptable step response. The values of the parameters considered in this paper both for the case $k_3 = 0$ and $k_3 \neq 0$ are shown in Table II.

IV. INVERSION-BASED FEEDFORWARD CONTROL DESIGN

It is well-known that a suitably designed feedforward control action is capable to significantly improve the performance of a control system. Here we propose to use a dynamic-inversion approach for determining the command input function $r(\cdot)$ to be applied to the closed-loop control system in order to obtain a predefined output transition from the value y_0 to the value y_1. A convenient choice for the desired output function is to use the so-called transition polynomial [4] which has the great advantage of easily addressing explicitly the constraint on the maximum and minimum acceleration of the mass. A transition polynomial is a polynomial function parameterised by the transition time τ, namely,

$$y_d(t;\tau) = y_0 + (y_1 - y_0)\frac{(2n+1)!}{h!\tau^{2n+1}}\sum_{i=0}^{n}\frac{(-1)^{n-i}}{i!(n-i)!(2n-i+1)}\tau^i t^{2n-i+1} \tag{12}$$

if $0 \le t \le \tau$, while $y_d(t;\tau) = y_0$ if $t < 0$ and $y_d(t;\tau) = y_1$ if $t > \tau$. It is worth noting that with this choice the desired output has neither undershooting nor overshooting and it is continuous until the nth order. The choice of the order n of the polynomial and of the output transition time τ derives directly from the procedure of finding the input function $r(t)$ that causes (in the nominal case) the desired output $y_d(t;\tau)$ and that is illustrated hereafter.

By substituting $y_d(t;\tau)$ in the state-space equations of the system (9) we obtain the desired trajectory of the states and the desired control variable $v(t)$ as (hereafter we often drop the arguments of the functions for the sake of simplicity):

$$\begin{aligned} x_{1d} &= y_d \\ x_{2d} &= \dot{y}_d \\ x_{3d} &= \pm\frac{\sqrt{-2\,\lambda\,\mu\,(-g\,m+\dot{x}_{2d}\,m+b\,x_{2d})}(1+\mu\,x_{1d})}{\lambda\,\mu} \\ &= \pm\frac{\sqrt{-2\,\lambda\,\mu\,(-g\,m+\ddot{y}_d\,m+b\,\dot{y}_d)}(1+\mu\,y_d)}{\lambda\,\mu} \end{aligned} \tag{13}$$

$$\begin{aligned} v_d &= \\ &= \frac{R\,x_{3d}+R\,x_{3d}\,\mu^2 {x_{1d}}^2+2\,R\,x_{3d}\,\mu\,x_{1d}-\mu\,x_{2d}\,x_{3d}\,\lambda+\dot{x}_{3d}\,\lambda+\dot{x}_{3d}\,\lambda\,\mu\,x_{1d}}{(1+\mu\,x_{1d})^2} \\ &= \frac{R\,x_{3d}+R\,x_{3d}\,\mu^2 {y_d}^2+2\,R\,x_{3d}\,\mu\,y_d-\mu\,\dot{y}_d\,x_{3d}\,\lambda+\dot{x}_{3d}\,\lambda+\dot{x}_{3d}\,\lambda\,\mu\,y_d}{(1+\mu\,y_d)^2} \end{aligned} \tag{14}$$

Note that the positive solution for x_{3d} (namely, the desired current to be applied to the electromagnet) has to be retained because of the constraint explained in Section III. Then, by applying (14) in (10) we obtain

$$x_{4d} = -\frac{v_d + k_{11}\,x_{1d} + k_{12}\,x_{2d} + k_{13}\,x_{3d} + k_3\,r_d - k_3\,x_{1d}}{k_2} \tag{15}$$

Finally, by taking into account that $\dot{x}_{4d} = r_d - y_d$ we can write

$$\dot{x}_{4d} = \frac{-\dot{v}_d - k_{11}\,x_{2d} - k_{12}\,\dot{x}_{2d} - k_{13}\,\dot{x}_{3d} - k_3\,\dot{r}_d + k_3\,x_{2d}}{k_2} \tag{16}$$

In order to determine the command input function $r_d(t;\tau)$ it is worth at this point considering the two cases $k_3 = 0$ and $k_3 \neq 0$.

Case 1: $k_3 = 0$

By considering $k_3 = 0$ in (16) and by considering $\dot{x}_{4d} = r_d - y_d$ we obtain

$$r_d = \frac{-\dot{v}_d - k_{11}\,x_{2d} - k_{12}\,\dot{x}_{2d} - k_{13}\,\dot{x}_{3d} + k_2\,x_{1d}}{k_2} \tag{17}$$

It is worth noting that, since $y_d = x_{1d}$ is constant for $t < 0$ and $t > \tau$, we have that in these time intervals it is $x_{2d} = 0$, $\dot{x}_{2d} = 0$, $\dot{x}_{3d} = 0$, $\dot{v}_d = 0$. In other words, the system attains a new equilibrium state at $t = \tau$. Then, the command signal to be applied to the closed-loop system is y_0 for $t < 0$, y_1 for $t > \tau$ and (17) for $0 \le t \le \tau$.

Remark 1. The state of the system can be represented by a combination of the output function and its derivatives until a fixed order. This actually means that the system is flat [6], [7].

Case 2: $k_3 \neq 0$

By considering $k_3 \neq 0$ in (16) and by considering again $\dot{x}_{4d} = r_d - y_d$ we obtain

$$\dot{r}_d = \frac{-\dot{v}_d - k_{11}\,x_{2d} - k_{12}\,\dot{x}_{2d} - k_{13}\,\dot{x}_{3d} + k_3\,x_{2d} + x_{1d}\,k_2 - r_d\,k_2}{k_3} \tag{18}$$

which can be rewritten as

$$\dot{r}_d = -r_d\frac{k_2}{k_3} + h + x_{1d}\frac{k_2}{k_3} \tag{19}$$

where

$$h = \frac{-\dot{u}_d - k_{11}\,x_{2d} - k_{12}\,\dot{x}_{2d} - k_{13}\,\dot{x}_{3d} + k_3\,x_{2d}}{k_3} \tag{20}$$

Thus, the expression of the command input r_d can be determined by solving the following differential equation

$$\dot{r}_d(t) = -r_d(t)\frac{k_2}{k_3} + h(t) + x_{1d}(t)\frac{k_2}{k_3} \tag{21}$$

with the initial condition

$$r_d(0) = y_0.$$

The solution is determined as

$$r_d(t) = e^{-k_2/k_3 t}\left[y_0 + \int_0^t \left(h(\zeta) + k_2/k_3 \cdot x_{1d}(\zeta)\right)\cdot e^{k_2/k_3 t} d\zeta\right] \tag{22}$$

It is worth stressing that for $t > \tau$ we have $h(t) = 0$, $x_{1d} = y_1$ and the resulting expression of the command input can be derived from (22) as

$$r_d(t) = r_d(\tau)\cdot e^{-k_2/k_3(t-\tau)} + y_1\cdot\left(1 - e^{-k_2/k_3(t-\tau)}\right). \tag{23}$$

Thus, the command input is not constant for $t > \tau$. By noting that $r(t)$ converges exponentially to y_1 for $t \to +\infty$ we can however truncate the synthesised function (23) with arbitrary precision [8]. Indeed, the command input exhibits a post-actuation time interval.

From the passages employed to derive the command input function, it appears that the fourth order time derivative of the desired output function is necessary to determine $r(t;\tau)$. Thus, $n = 4$ in (12) have to be selected in order to obtain a continuous command input function. We have therefore

$$y_d(t;\tau) = y_0 + (y_1 - y_0)\left(70\,\frac{t^9}{\tau^9} - 315\,\frac{t^8}{\tau^8} + 540\,\frac{t^7}{\tau^7} - 420\,\frac{t^6}{\tau^6} + 126\,\frac{t^5}{\tau^5}\right) \tag{24}$$

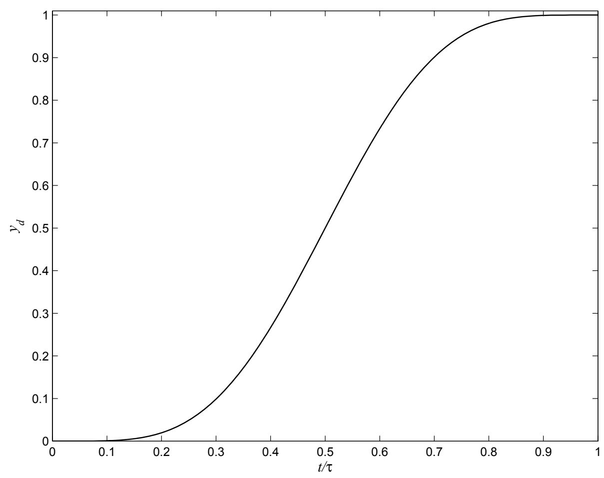

Fig. 3. Plot of the transition polynomial employed as a desired output function.

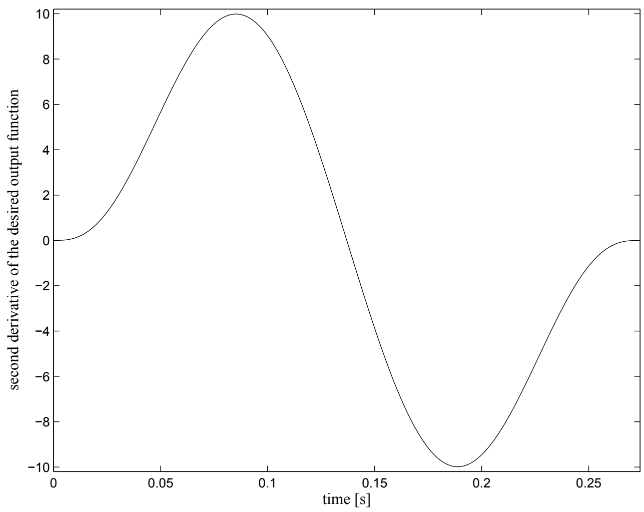

Fig. 4. Second derivative $\ddot{y}_d(t)$ of the desired output function for $\tau = 0.277$ s.

A plot of the transition polynomial (24) with $y_0 = 0$ and $y_1 = 1$ and with the time axis normalised by τ is shown in Figure 3. In this context, the transition time τ can be selected in order to satisfy the constraints on the maximum acceleration of the object which has not to be greater than the gravity acceleration constant g. Indeed, from (24) it can be derived that the maximum acceleration is [4]

$$\ddot{y}_{d,max} = \frac{1215}{343}\sqrt{7}\frac{(y_1 - y_0)}{\tau^2} \tag{25}$$

and we have therefore

$$\tau \geq \frac{3.064\,\sqrt{g\,(y_1 - y_0)}}{g}. \tag{26}$$

V. ILLUSTRATIVE RESULTS

A. Nominal Case

As an illustrative example, consider the control task where $y_0 = 0.02$ m and $y_0 = 0.1$ m. By applying expression (26) we have that the minimum transition time, which is actually selected for the control task, is $\tau = 0.277$ s. The second derivative of the determined output function, namely, the acceleration of the mass, is plotted in Figure 4 where it appears that it meets the given constraint. The determined command input functions for $k_3 = 0$ and $k_3 \neq 0$ are shown respectively in Figures 5 and 6, while the corresponding control variable (namely, the voltage applied to the electromagnet) is plotted in Figure 7 (note that it is the same for both cases). The system output is shown in Figure 8 where, for the sake of comparison, the response to a step command input is also shown (for both cases $k_3 = 0$ and $k_3 \neq 0$). It appears that, since we are in the nominal case, the desired output function is attained both for $k_3 = 0$ and $k_3 \neq 0$ and the use of an inversion-based command input allows to improve significantly the performance with respect to the use of a traditional step signal.

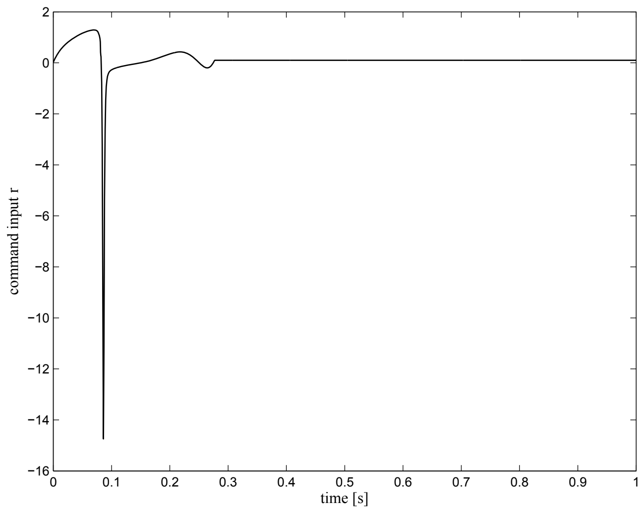

Fig. 5. Command function for $k_3 = 0$.

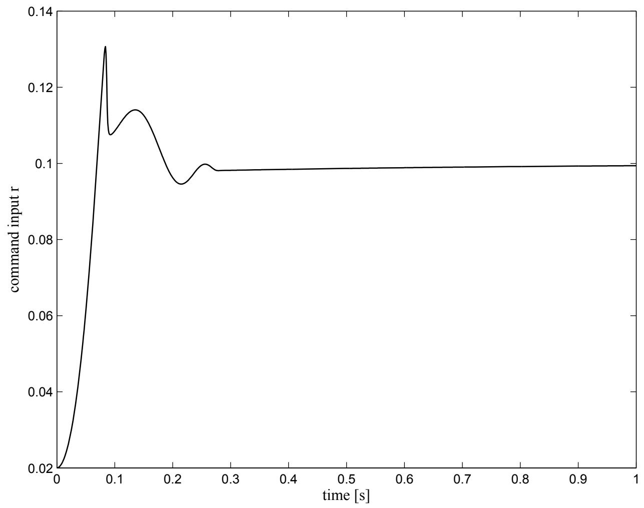

Fig. 6. Command function for $k_3 \neq 0$.

TABLE III

RESULTS OBTAINED BY CONSIDERING A PERTURBED SYSTEM.

Control system	Variation of 50% for the mass value m		Variation of 5% for the mass value m		Variation of 5% for parameters m, λ, R, μ		Variation of 1% for parameters m, λ, R, μ	
	T_s	$O(\%)$	T_s	$O(\%)$	T_s	$O(\%)$	T_s	$O(\%)$
feedback $k_3 = 0$	1.55	12.7	0.98	3.2	1.38	11.0	0.99	3.9
feedback $k_3 \neq 0$	1.74	5.7	0.71	2.5	1.31	3.9	0.73	2.5
feedforward-feedback $k_3 = 0$	1.40	13.4	1.17	3.7	1.44	11.2	1.18	4.3
feedforward-feedback $k_3 \neq 0$	1.45	7.2	0.31	0.8	1.01	3.6	0.24	0.7

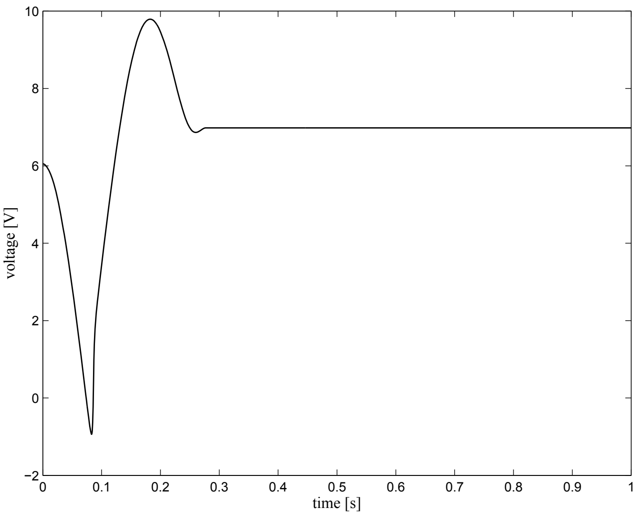

Fig. 7. Control variable for $k_3 = 0$ and $k_3 \neq 0$.

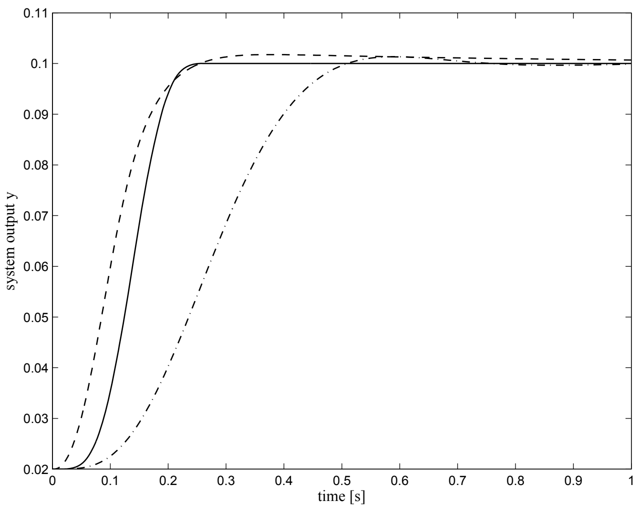

Fig. 8. System output with the inversion-based command input (solid line) and with a step command input for $k_3 = 0$ (dash-dot line) and for $k_3 \neq 0$ (dashed line).

B. Robustness Analysis

Being a feedforward approach (though applied to a closed-loop control system that reduces the effects of the system uncertainties), the performance achieved by applying the inversion-based command input obviously relies on the accuracy of the system model. In order to verify the effectiveness of the method in the presence of structured (parametric) uncertainties, the command input determined by considering the nominal model has been applied to the closed-loop system where the values of the system parameters have been modified. In particular, we considered a perturbation of 50% and 5% for the mass of the object, and a 5% and 1% perturbation for all the parameters. The worst-case results, obtained with the feedback controller only and with the feedforward/feedback controller with $k_3 = 0$ and $k_3 \neq 0$ in both cases, are summarised in Table III in terms of maximum percentage overshoot O and 1% settling time T_s. It appears that in general it is convenient to add the feedback term proportional to the control error (namely, it is worth setting $k_3 \neq 0$) and in this context the use of the inversion-based feedforward action allows to significantly improve the performance with respect to the use of a step set-point signal. Moreover, the command signal for the case $k_3 = 0$ exhibits a deep spike that is not present in the command signal for the case $k_3 \neq 0$ and this makes it less suitable for a practical implementation. This is due to the fact that without the term proportional to the control error the required fast variation of the output of the integrator is possible only by applying a much faster variation to its input (see Figures 5 and 6).

VI. CONCLUSIONS

In this paper we have proposed a feedforward/feedback approach for a magnetic levitation system. The considered testbed allows to address many issues that have to be addressed carefully in the design of a control system and it is particularly useful for educational purposes. In particular, we pointed out that the presence of a term which is proportional to the control error, in addition to the inclusion of the integral action, allows to improve significantly the performance of the state-feedback controller. Then, the use of an inversion-based feedforward action allows to further increase the control performance. In this context, the use of a transition polynomial as a desired output function plays a key role in addressing the system constraints simply. It is believed that the proposed design methodology can be exploited also for other nonlinear constrained control problems.

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