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Generalized bang–bang control for feedforward constrained regulation ☆

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ABSTRACT

In the behavioral framework for continuous-time linear scalar systems, simple sufficient conditions for the solution of the minimum-time rest-to-rest feedforward constrained control problem are provided. The investigation of the time-optimal input–output pair reveals that the input or the output saturates on the assigned constraints at all times except for a set of zero measure. The resulting optimal input is composed of sequences of bang–bang functions and linear combinations of the modes associated to the zero dynamics. This signal behavior constitutes a generalized bang–bang control that can be fruitfully exploited for feedforward constrained regulation. Using discretization, an arbitrarily good approximation of the optimal generalized bang–bang control is found by solving a sequence of linear programming problems. Numerical examples are included.

1. Introduction

Recently in the control engineering literature, it has been emphasized that to achieve high performances in real applications, due attention has to be paid to the constraints which all the plant variables must comply with. In particular, the main approaches to control system design with input and output constraints are the following:

- Antiwindup and override feedback schemes. This is the standard approach in the practical industrial context; see, for example, the recent book of Glattfelder and Schaufelberger (2003).
- Model predictive control. In the receding horizon strategy, input constraints as well as output ones can be naturally considered in designing the feedback controller; see, for instance, Maciejowski (2002).

In this paper we address the subject of controlling a continuous-time scalar linear system with input and output constraints by setting a purely feedforward regulation problem to be solved in minimum-time. We assume that the system is stable and want to find a minimum-time feedforward input that brings the system from a current rest condition to a new desired rest condition while satisfying at all times given amplitude constraints on the input and the output. In such a way, we can naturally deal with both actuator limitations and overshooting and undershooting requirements.

It is well known that the minimum-time feedforward control with input constraints only is given by the so-called bang–bang control, i.e. the input signal switches between its extreme allowed values (Lewis & Syrmos, 1995). In a behavioral setting, this paper shows that in the presence of both input and output constraints the minimum-time input–output pair enjoys the property that the saturation of the input or the output signal occurs almost everywhere. Therefore, the optimal feedforward input is given by a sequence of bang–bang functions and linear combinations of the system zero modes. This type of optimal control can be viewed as a *generalized bang–bang control*. For the actual computation of this time-optimal control, the proposed idea is to discretize the continuous-time system and to solve the resulting discrete-time problem by means of linear programming. In fact, in the discrete-time case, input and output constraints can be represented as linear inequalities and the minimum number of steps needed for a rest-to-rest transition can be found with a sequence of linear feasibility tests.

The idea of using linear programming for solving a minimum-time problem for linear discrete-time systems subject to amplitude input constraints dates back to Zadeh (1962). Subsequently, various contributions have appeared by focusing on some improvements for this discrete-time problem (Bashein, 1971; Kim & Engell, 1994; Scott, 1986). In this paper, we prove that the optimal

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discrete-time solution converges to the optimal continuous-time one when the sampling time approaches zero.

This article is structured as follows. In the second section the problem of minimum-time feedforward constrained regulation is presented for linear continuous-time systems. This is done in the framework of the behavioral approach (see Polderman and Willems (1998)). Herein the main result is a simple sufficient condition (Theorem 1) that guarantees the problem solvability. The third section is devoted to the study of the structure of the time-optimal solution. By exploiting the convexity of the system accessible set the main result (Theorem 2) is deduced. It states that the optimal input–output pair saturate on extreme values almost everywhere. From a corollary of this theorem (Corollary 1) the optimal feedforward input is then characterized as a generalized bang–bang control. In the fourth section the minimum-time constrained problem is introduced for discrete-time systems. A feasibility test is presented in Proposition 3 which is followed by an algorithm that computes the optimal discrete-time control through the solutions of a sequence of linear programming problems. Section 5 presents a convergence result. Theorem 4 shows that the optimal solution for the discretized system converges to the solution of the original continuous-time system as the sampling time goes to zero. Some simulation results are presented in Section 6. Conclusions are reported in Section 7.

Notation. C^i denotes the set of real functions defined over $\mathbb{R}$ that are continuous till the ith derivative. The ith order differential operator is D^i. The L_∞ norm of a real function $f(t)$ defined and bounded over $\mathbb{R}$ is $\|f(\cdot)\|_\infty := \sup_{t\in\mathbb{R}} |f(t)|$ and the L_1 norm is $\|f(\cdot)\|_1 := \int_{-\infty}^{+\infty} |f(t)|\mathrm{d}t$. Given $x \in \mathbb{R}$, $\lceil x\rceil = \min\{z \in \mathbb{Z} : z \geq x\}$, $\lfloor x\rfloor = \max\{z \in \mathbb{Z} : z \leq x\}$. Given a subset $\mathcal{S} \subset \mathbb{R}^n$, $\partial\mathcal{S}$ denotes the boundary of $\mathcal{S}$, $\mathrm{cl}(\mathcal{S})$ is the closure of $\mathcal{S}$. If $\mathcal{S} \subset \mathbb{R}^n$ is a Lebesgue measurable set then $|\mathcal{S}|$ denotes its measure. The space of locally integrable real functions is denoted by L_1^{loc}.

2. The minimum-time feedforward constrained regulation problem

Consider a linear, stable, continuous-time system Σ described by the scalar, strictly proper transfer function

$$H(s) = \frac{b(s)}{a(s)} = \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_0}. \tag{1}$$

The system static gain is $H(0) = \frac{b_0}{a_0} \neq 0$ and the input and output are denoted by u and y respectively. Also, assume that polynomials $a(s)$ and $b(s)$ are coprime with $b_m \neq 0$ and $m < n$. With $h(t)$ we denote the impulse response of system Σ, i.e. $h(t) = \mathcal{L}^{-1}[H(s)]$ where $\mathcal{L}^{-1}$ denotes the inverse Laplace transform. The behavior set of Σ can be introduced as the set $\mathcal{B}$ of all input–output pairs $(u(\cdot), y(\cdot)) \in L_1^{\mathrm{loc}} \times \in L_1^{\mathrm{loc}}$ that are "weak" solutions of the differential equation (Polderman & Willems, 1998) ($a_n := 1$):

$$\sum_{i=0}^{n} a_i D^i y = \sum_{i=0}^{m} b_i D^i u. \tag{2}$$

The control aim is to find a minimum-time feedforward input that causes a rest-to-rest transition from $y = 0$ to $y = y_f$ subject to arbitrarily assigned input and output constraints ($y_f \in \mathbb{R}$ is any desired output value). The rest condition of Σ is characterized by the set of input–output equilibrium points designated as $\mathcal{E} := \{(u, y) \in \mathbb{R}^2 : y = H(0)u\}$. We introduce, as a special subset of $\mathcal{B}$, the set $\mathcal{T}_{\mathbf{p}}$ of all rest-to-rest transitions from $(0, 0) \in \mathcal{E}$ to $(\frac{y_f}{H(0)}, y_f) \in \mathcal{E}$ subject to input and output constraints.

Definition 1. Let be given a constraint parameter set $\mathbf{p} := \{U_c, Y_c, y_f\}$ where $U_c = [u_c^-, u_c^+]$ and $Y_c = [y_c^-, y_c^+]$ are the constraint intervals for the input and output respectively and y_f is the final output rest value for which

$$\left\{0, \frac{y_f}{H(0)}\right\} \subset U_c \quad \text{and} \quad \{0, y_f\} \subset Y_c. \tag{3}$$

Then define $\mathcal{T}_{\mathbf{p}}$ as the set of all pairs $(u(\cdot), y(\cdot)) \in \mathcal{B}$ for which there exists $t_f > 0$ such that:

$$u(t) = 0 \quad \forall t < 0, \qquad u(t) = \frac{y_f}{H(0)} \quad \forall t \geq t_f, \tag{4}$$

$$u(t) \in U_c \quad \forall t \in [0, t_f], \tag{5}$$

$$y(t) = 0 \quad \forall t < 0, \qquad y(t) = y_f \quad \forall t \geq t_f, \tag{6}$$

$$y(t) \in Y_c \quad \forall t \in [0, t_f]. \tag{7}$$

The constraints intervals introduced in the above definition can encapsulate all the typical amplitude limitations that apply to the input and the output for any set-point regulation problem. For example, if a regulation problem requires $|u(t)| \leq u_{MAX}$, $\forall t \in \mathbb{R}$, a maximum 10% overshooting and 5% undershooting on the output signal we can assign (consider $y_f > 0$): $U_c = [-u_{MAX}, +u_{MAX}]$, $Y_c = [-0.05y_f, +1.1y_f]$.

Lemma 1. *Given system Σ (1) and any $T > 0$, there exist two positive constants M_u, M_y such that for any vector $\mathbf{w} = [w_0, w_1, \ldots, w_{n-1}]^T \in \mathbb{R}^n$ and any $a \in \mathbb{R}$, there exists an input–output pair $(\hat{u}(\cdot), \hat{y}(\cdot)) \in \mathcal{B} \cap C^n$ such that*

(1) $\hat{u}(t) = 0, \forall t \in (-\infty, a] \cup [a + T, +\infty)$;
(2) $\hat{y}(t) = 0$, $\forall t \leq a$, $\hat{y}(a + T) = w_0$, $D\hat{y}(a + T) = w_1, \ldots, D^{n-1}\hat{y}(a + T) = w_{n-1}$;
(3) $\|\hat{u}(\cdot)\|_\infty \leq M_u\|\mathbf{w}\|$, $\|\hat{y}(\cdot)\|_\infty \leq M_y\|\mathbf{w}\|$.

Proof. Set $g_i(t) = h^{(i)}(T - t)t^n(T - t)^n$. Functions $g_0(t)$, $g_1(t), \ldots, g_{n-1}(t)$ are linearly independent, therefore the following Gramian matrix is nonsingular

$$\mathbf{G} = \int_0^T \begin{bmatrix} g_0(t) \\ g_1(t) \\ \vdots \\ g_{n-1}(t) \end{bmatrix} [g_0(t), g_1(t), \ldots, g_{n-1}(t)]\mathrm{d}t.$$

Define the input

$$u(t) = \begin{cases} 0 & \text{if } t \in (-\infty, 0] \cup [T, +\infty) \\ t^n(T - t)^n[g_0(t), g_1(t), \ldots, g_{n-1}(t)] \cdot \mathbf{G}^{-1}\mathbf{w} \\ \quad \text{if } 0 \leq t \leq T. \end{cases}$$

This input signal belongs to C^n and satisfies $D^i u(0) = D^i u(T) = 0$, $i = 0, \ldots, n - 1$. Define the output as follows

$$y(t) = \begin{cases} 0 & \text{if } t \leq 0 \\ \displaystyle\int_0^t h(t - \tau)u(\tau)\mathrm{d}\tau & \text{if } t \geq 0. \end{cases}$$

Clearly $y \in C^n$, and

$$\begin{aligned} y(T) &= \int_0^T h(T - \tau)u(\tau)\mathrm{d}\tau \\ &= \int_0^T g_0(\tau)[g_0(\tau), g_1(\tau), \ldots, g_{n-1}(\tau)]\mathrm{d}\tau\mathbf{G}^{-1}\mathbf{w} \\ &= [1, 0, \ldots, 0]\mathbf{w} = w_0. \end{aligned}$$

Moreover $Dy(T) = \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t h(t - \tau)u(\tau)\mathrm{d}\tau\Big|_{t=T} = \int_0^T Dh(T - \tau) u(\tau)\mathrm{d}\tau + h(0)u(T)$, and considering $u(T) = 0$, then $Dy(T) = \int_0^T g_1(\tau)[g_0(\tau), g_1(\tau), \ldots, g_{n-1}(\tau)]\mathrm{d}\tau\mathbf{G}^{-1}\mathbf{w} = [0, 1, \ldots, 0]\mathbf{w} = w_1$. Repeating the same procedure it follows that $D^i y(T) = w_i$, $i = 2, \ldots, n - 1$. Now define by time translation $\hat{u}(t) = u(t - a)$ and $\hat{y}(t) = y(t - a)$ so that $(\hat{u}, \hat{y}) \in \mathcal{B} \cap C^n$ and statements (1) and (2) of Lemma 1 are evidently verified. Finally, statement (3) holds since

$$\|\hat{u}\|_\infty = \|u\|_\infty \leq T^{4n}\left(\|h\|_\infty + \|Dh\|_\infty + \cdots + \|D^{n-1}h\|_\infty\right)\|\mathbf{G}^{-1}\|\|\mathbf{w}\|,$$
$$\|\hat{y}\|_\infty = \|y\|_\infty \leq \|h\|_1\|u\|_\infty$$

where $\|h\|_1 = \int_0^{+\infty}|h(v)|\mathrm{d}v$ is the peak gain of system Σ. □

The following theorem gives a straightforward sufficient condition to ensure that $\mathcal{T}_\mathbf{p}$ is not empty.

Theorem 1. *Set $\mathcal{T}_\mathbf{p}$ is not empty if*

$$\left\{0, \frac{y_f}{H(0)}\right\} \subset (u_c^-, u_c^+) \quad \textit{and} \quad \{0, y_f\} \subset (y_c^-, y_c^+). \tag{8}$$

Proof. Without loss of generality we assume $H(0) > 0$ and $y_f > 0$. Let $l(t)$ be any C^n function such that

$$l(t) = 0 \quad \forall t < 0, \qquad l(t) = \frac{y_f}{H(0)} \quad \forall t > 1,$$
$$0 \leq l(t) \leq \frac{y_f}{H(0)} \quad \forall t \in [0, 1].$$

Given a real constant $\epsilon > 0$, let the input to system Σ be given by $l(\epsilon t)$ and denote by $y(t;\epsilon)$ the corresponding output with $y(t;\epsilon) = 0\ \forall t < 0$. Hence, the following limit holds:

$$\lim_{\epsilon\to 0}\|y(t;\epsilon) - l(\epsilon t)H(0)\|_\infty = 0. \tag{9}$$

Indeed, the Laplace transform of $y(t;\epsilon) - l(\epsilon t)H(0)$ is given by:

$$L(s;\epsilon)\,(H(s) - H(0)), \tag{10}$$

where $L(s;\epsilon) := \mathcal{L}[l(\epsilon t)]$. Since $H(s) - H(0) = s\tilde{H}(s)$, where $\tilde{H}(s)$ is a suitable stable biproper transfer function, expression (10) can be written as $L(s;\epsilon)s\tilde{H}(s) = \tilde{H}(s)\mathcal{L}\left[\frac{\mathrm{d}}{\mathrm{d}t}l(t\epsilon)\right]$. Therefore

$$\|y(t;\epsilon) - l(\epsilon t)H(0)\|_\infty \leq \int_0^{+\infty}|\tilde{h}(v)|\mathrm{d}v \cdot \left\|\frac{\mathrm{d}}{\mathrm{d}t}l(t\epsilon)\right\|_\infty,$$

where $\tilde{h}(t) = \mathcal{L}^{-1}[\tilde{H}(s)]$ and $\int_0^{+\infty}|\tilde{h}(v)|\mathrm{d}v$ is the peak gain of $\tilde{H}(s)$. Since $\left\|\frac{\mathrm{d}}{\mathrm{d}t}l(t\epsilon)\right\|_\infty = \epsilon\left\|\frac{\mathrm{d}}{\mathrm{d}(t\epsilon)}l(t\epsilon)\right\|_\infty$, limit (9) is proved. Moreover, from $\frac{\mathrm{d}^i}{\mathrm{d}t^i}l(t\epsilon) = \epsilon^i \cdot \frac{\mathrm{d}^i}{\mathrm{d}(t\epsilon)^i}l(t\epsilon)$ we have

$$\lim_{\epsilon\to 0}\left\|\frac{\mathrm{d}^i}{\mathrm{d}t^i}l(t\epsilon)\right\|_\infty = 0, \quad i = 1,\ldots,n-1. \tag{11}$$

Then, using again the peak gain concept it follows that

$$\lim_{\epsilon\to 0}\left\|\frac{\mathrm{d}^i}{\mathrm{d}t^i}y(t;\epsilon)\right\|_\infty = 0, \quad i = 1,\ldots,n-1. \tag{12}$$

So far we have constructed a family of input–output pairs $(l(\epsilon t), y(t;\epsilon)) \in \mathcal{B}$ parameterized by $\epsilon > 0$. Now, consider $\epsilon < 1$ and choose the "correcting pair" $(\hat{u}(t;\epsilon), \hat{y}(t;\epsilon)) \in C^n \cap \mathcal{B}$ accordingly to Lemma 1 with $a = \epsilon^{-1} - 1$, $T = 1$ and

$$\hat{u}(t;\epsilon) = 0, \quad \text{for } t \in (-\infty, \epsilon^{-1} - 1] \cup [\epsilon^{-1}, +\infty),$$
$$\hat{y}(\epsilon^{-1};\epsilon) = y_f - y(\epsilon^{-1};\epsilon),$$
$$\frac{\mathrm{d}^i}{\mathrm{d}t^i}\hat{y}(t;\epsilon)|_{t=\epsilon^{-1}} = -\frac{\mathrm{d}^i}{\mathrm{d}t^i}y(t;\epsilon)|_{t=\epsilon^{-1}}, \quad i = 1,\ldots,n-1.$$

Therefore the pair $(\bar{u}(t;\epsilon), \bar{y}(t;\epsilon)) \in C^n \cap \mathcal{B}$ defined by

$$(\bar{u}(t;\epsilon), \bar{y}(t;\epsilon)) = (l(t\epsilon), y(t;\epsilon)) + (\hat{u}(t;\epsilon), \hat{y}(t;\epsilon)),$$

satisfies the rest conditions at time $t = \epsilon^{-1}$:

$$\bar{u}(\epsilon^{-1};\epsilon) = y_f H(0)^{-1}, \qquad \frac{\mathrm{d}^i}{\mathrm{d}t^i}\bar{u}(t;\epsilon)|_{t=\epsilon^{-1}} = 0, \quad i = 1,\ldots,m,$$
$$\bar{y}(\epsilon^{-1};\epsilon) = y_f, \qquad \frac{\mathrm{d}^i}{\mathrm{d}t^i}\bar{y}(t;\epsilon)|_{t=\epsilon^{-1}} = 0, \quad i = 1,\ldots,n-1; \tag{13}$$

hence $\bar{y}(t;\epsilon) = y_f$, $\forall t \geq \epsilon^{-1}$. Because of (9) and (12) and statement (3) of Lemma 1, it follows that

$$\lim_{\epsilon\to 0}\|\hat{u}(t;\epsilon)\|_\infty = 0, \tag{14}$$

$$\lim_{\epsilon\to 0}\|\hat{y}(t;\epsilon)\|_\infty = 0. \tag{15}$$

By virtue of (8), $|u_c^-| > 0$ and $u_c^+ - \frac{y_f}{H(0)} > 0$ so that there exists $\epsilon_u > 0$ such that, by (14), $\forall\epsilon < \epsilon_u$

$$\|\hat{u}(t;\epsilon)\|_\infty \leq \min\left\{|u_c^-|, u_c^+ - \frac{y_f}{H(0)}\right\}.$$

By virtue of (8), $|y_c^-| > 0$ and $y_c^+ - y_f > 0$ and by (9) there exists $\epsilon_{y1} > 0$ such that $\forall\epsilon < \epsilon_{y1}$

$$\|y(t;\epsilon) - l(\epsilon t)H(0)\|_\infty \leq \frac{\min\{|y_c^-|, y_c^+ - y_f\}}{2}$$

and by (15) there exists $\epsilon_{y2} > 0$ such that $\forall\epsilon < \epsilon_{y2}\|\hat{y}(t;\epsilon)\|_\infty \leq \frac{\min\{|y_c^-|, y_c^+ - y_f\}}{2}$. Finally setting $\epsilon_0 = \min\{\epsilon_u, \epsilon_{y1}, \epsilon_{y2}\}$ we obtain that $(\bar{u}(t;\epsilon_0), \bar{y}(t;\epsilon_0)) \in \mathcal{T}_\mathbf{p}$. □

Remark 1. Note that sufficient condition (8) differs from assumption (3) of Definition 1 defining $\mathcal{T}_\mathbf{p}$ just for the exclusion of the four endpoints of intervals U_c and Y_c. Hence, condition (8) implies that there exists at least a small distance between the constraints extrema and the corresponding steady-state input–output values. This permits to construct (as shown in the proof) an input–output pair that reaches the steady-state in finite time while respecting the constraints.

Once inclusions (8) are satisfied, the emerging natural problem is to determine among all the constrained transitions of $\mathcal{T}_\mathbf{p}$ the fastest one, i.e. the optimal rest-to-rest transition with associated minimum transition time t_f^*:

$$t_f^* := \inf_{(u(\cdot),y(\cdot))\in\mathcal{T}_\mathbf{p}} T_f(u(\cdot), y(\cdot)) \tag{16}$$

where T_f is the following functional

$$T_f(u(\cdot), y(\cdot)) = \inf\left\{t_1 : u(t) = \frac{y_f}{H(0)}, y(t) = y_f, \forall t \geq t_1\right\} \tag{17}$$

which is well defined by Definition 1. Note that t_f^* corresponds to the minimum $T_f(u(\cdot), y(\cdot))$ that is achievable with an optimal pair $(u^*(\cdot), y^*(\cdot))$ that is essentially unique in $\mathcal{T}_\mathbf{p}$ (proofs are reported in Appendix A).

On the other hand, from a control viewpoint the problem is to directly determine the optimal feedforward input $u^*(t)$ that corresponds to minimum-time t_f^*. An approximate solution to this problem using linear programming is exposed in Section 5.

3. Characterization of the time-optimal solution for the continuous-time case

This section gives a characterization of the time-optimal solution $(u^*(\cdot), y^*(\cdot)) \in \mathcal{T}_\mathbf{p}$ to the constrained set-point regulation problem for continuous-time systems.

Definition 2. Consider a linear system of the form

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u, \qquad y = \mathbf{c}\mathbf{x}, \tag{18}$$

where $\mathbf{x} \in \mathbb{R}^n$, $u \in \mathbb{R}$, $y \in \mathbb{R}$ and intervals U_c, Y_c are given constraints on the input and the output respectively. Then, the *constrained reachable set* at final time T, starting from initial state $\mathbf{x}_0$ is denoted by $\mathcal{A}_{U_c,Y_c}(\mathbf{x}_0, T)$ and is defined as

$$\Big\{\mathbf{x}_1 \in \mathbb{R}^n : \mathbf{x}_1 = e^{\mathbf{A}T}\mathbf{x}_0 + \int_0^T e^{\mathbf{A}(T-\tau)}\mathbf{b}u(\tau)d\tau,$$
$$u \in L_1^{\text{loc}}, u(t) \in U_c, y(t) \in Y_c, \forall t \in [0, T]\Big\},$$

i.e. $\mathcal{A}_{U_c,Y_c}(\mathbf{x}_0, T)$ is the set of all states that can be reached from x_0 at time T while satisfying the given input and output constraints.

Proposition 1. *For any $\mathbf{x}_0 \in \mathbb{R}^n$ and $T > 0$ the set $\mathcal{A}_{U_c,Y_c}(\mathbf{x}_0, T)$ is convex.*

Proof. The result is a straightforward consequence of the linearity of system (18). □

When output constraints are not present ($Y_c = \mathbb{R}$) then, by the classical bang–bang theory, the following proposition holds; its proof can be found in Jurdjevic (1997, p. 302).

Proposition 2. *Assume that system (18) is controllable, then the control u^* that drives the system from the initial state $\mathbf{x}_0$ to the final state $\mathbf{x}_1$ in minimum-time t^* with the input constraint $u^*(t) \in U_c = [u_c^-, u_c^+]$, $\forall t \in [0, t^*]$, is such that $u^*(t) \in \{u_c^-, u_c^+\}$, almost everywhere in $[0, t^*]$.*

The following theorem is the main result of this section and characterizes the time-optimal solution with input and output constraints, extending Proposition 2.

Theorem 2. *Given the time-optimal pair $(u^*(\cdot), y^*(\cdot)) \in \mathcal{T}_{\mathbf{p}}$, the set*

$$\mathcal{S} = \left\{t \in [0, t_f^*] : u^*(t) \notin \{u_c^-; u_c^+\}, \text{ and } y^*(t) \notin \{y_c^-; y_c^+\}\right\} \quad (19)$$

has null (Lebesgue) measure.

Proof. Let system Σ (1), be represented by a controllable and observable realization of the form (18). The initial state is given by $\mathbf{x}(0) = \mathbf{0}$ and the final state at time t_f^* is given by $\mathbf{x}(t_f^*) = -\mathbf{A}^{-1}\mathbf{b}\frac{y_f}{H(0)}$.

Assume by contradiction that $|\mathcal{S}| \neq 0$. By Lebesgue integration theory, there exists a finite set of closed intervals $\mathcal{I}_i$, $i = 1, \ldots, n_s$ such that

$$\mathcal{S} \supset \bigcup_i \mathcal{I}_i, \qquad \sum_i |\mathcal{I}_i| \neq 0, \quad (20)$$

in particular there exists an integer l such that $\mathcal{I}_l = [a, b]$, with $b - a > 0$ and $\forall t \in \mathcal{I}_l$, $u^*(t) \in (u_c^-, u_c^+)$, $y^*(t) \in (y_c^-, y_c^+)$. Thus there exists $\delta > 0$ such that $u^*(t) \in (u_c^- + \delta, u_c^+ - \delta)$, $y^*(t) \in (y_c^- + \delta, y_c^+ - \delta)$, $\forall t \in \mathcal{I}_l$. By the principle of optimality, state $\mathbf{x}_b := \mathbf{x}(b)$ belongs to the boundary of the constrained reachable set from $\mathbf{x}_a := \mathbf{x}(a)$ after a time $b-a$, that is $\mathbf{x}_b \in \partial\mathcal{A}_{U_c,Y_c}(\mathbf{x}_a, b-a)$. By Proposition 1, $\mathcal{A}_{U_c,Y_c}(\mathbf{x}_a, b-a)$ is a convex set, therefore the supporting hyperplane at $\mathbf{x}_b$, defined by $\{\mathbf{x} \in \mathbb{R}^n : p + \mathbf{q}^T\mathbf{x} = 0\}$ with

$$p + \mathbf{q}^T\mathbf{x}_b = 0, \quad (21)$$

satisfies the inequality

$$p + \mathbf{q}^T\mathbf{x} \leq 0 \quad \forall \mathbf{x} \in \mathcal{A}_{U_c,Y_c}(\mathbf{x}_a, b-a). \quad (22)$$

Hence, for any function $u(t) : [a, b] \to \mathbb{R}$ such that $u(t) \in U_c$ and $y(t) \in Y_c$, $\forall t \in [a, b]$ it follows that

$$p + \mathbf{q}^T\left\{e^{\mathbf{A}(b-a)}\mathbf{x}_a + \int_a^b e^{\mathbf{A}(b-t)}\mathbf{b}u(t)dt\right\} \leq 0 \quad (23)$$

where the equality holds for the optimal control $u^*(t)$. Choose any $\tilde{\mathbf{x}}_b \in \mathbb{R}^n$ satisfying

$$p + \mathbf{q}^T\tilde{\mathbf{x}}_b > 0. \quad (24)$$

Disregarding input and output constraints, there exists a control $\tilde{u}$ that drives the state from $\mathbf{x}_a$ to $\tilde{\mathbf{x}}_b$ in time $b - a$ by virtue of system controllability. Consider the linear combination $u_\lambda = (1-\lambda)u^* + \lambda\tilde{u}$. By linearity, the final state reached with control u_λ is given by $(1-\lambda)\mathbf{x}_b + \lambda\tilde{\mathbf{x}}_b$. By continuity, there exists a sufficiently small $\lambda_0 \in (0, 1)$, such that both the input and the corresponding output satisfy the constraints, that is $u_{\lambda_0}(t) \in U_c$, $y_{\lambda_0}(t) \in Y_c$. By (21) and (24), the final state $(1-\lambda_0)\mathbf{x}_b + \lambda_0\tilde{\mathbf{x}}_b$ reached with input u_{λ_0} satisfies the inequality $p + \mathbf{q}^T[(1-\lambda_0)\mathbf{x}_b + \lambda_0\tilde{\mathbf{x}}_b] > 0$, which contradicts (22). □

Denote by $m_1^P(t), m_2^P(t), \ldots, m_n^P(t)$ the modes of pole dynamics of Σ and by $m_1^Z(t), m_2^Z(t), \ldots, m_m^Z(t)$ the modes of zero dynamics of Σ. A straightforward consequence of Theorem 2 is the following corollary that discloses the structure of the optimal pair (u^*, y^*).

Corollary 1. *There exist open, nonempty, nonoverlapping intervals I_i, $O_j \subset \mathbb{R}$, $i, j \in \mathbb{N}$ and real coefficients $\alpha_{0i}, \alpha_{1i}, \ldots, \alpha_{ni}, \beta_{0j}, \beta_{1j}, \ldots, \beta_{mj}$ such that*

1.$[0, t_f^*] = \bigcup_i \text{cl}(I_i) \cup \bigcup_j \text{cl}(O_j)$;

2. u^* and y^* are respectively a constant and a nonconstant function over interval I_i according to:

$$u^*(t) = u_c^- \quad \forall t \in I_i \quad \text{or} \quad u^*(t) = u_c^+ \quad \forall t \in I_i,$$

$$y^*(t) = \alpha_{0i} + \sum_{k=1}^{n} \alpha_{ki} m_k^P(t) \quad \forall t \in I_i; \quad (25)$$

3. u^* and y^* are respectively a nonconstant and a constant function over interval O_j according to:

$$u^*(t) = \beta_{0j} + \sum_{l=1}^{m} \beta_{lj} m_l^Z(t) \quad \forall t \in O_j, \quad (26)$$

$$y^*(t) = y_c^- \quad \forall t \in O_j \quad \text{or} \quad y^*(t) = y_c^+ \quad \forall t \in O_j.$$

Proof. From Theorem 2 there exist open, nonempty, nonoverlapping intervals I_i, $O_j \subset \mathbb{R}$, $i, j \in \mathbb{N}$ such that statement 1 is satisfied and $u^*(t) = u_c^-$ $\forall t \in I_i$ or $u^*(t) = u_c^+$ $\forall t \in I_i$ and $y^*(t) = y_c^-$ $\forall t \in O_i$ or $y^*(t) = y_c^+$ $\forall t \in O_i$. Hence, there exist real coefficients $\alpha_{0i}, \alpha_{1i}, \ldots, \alpha_{ni}, \beta_{0j}, \beta_{1j}, \ldots, \beta_{mj}$ such that relations (25) and (26) are verified. Functions $y^*(t)$ over I_i and $u^*(t)$ over O_i are actually nonconstant functions, i.e., $[\alpha_{1i}, \alpha_{2i}, \ldots, \alpha_{ni}]^T \neq 0$ for all i and $[\beta_{1j}, \beta_{2j}, \ldots, \beta_{nj}]^T \neq 0$ for all j. Indeed, by contradiction assume that $y^*(t)$ is a constant function over I_i. This means that pair (u^*, y^*) is at the equilibrium over I_i with $y^*(t) = H(0)u_c^-$ $\forall t \in I_i$ or $y^*(t) = H(0)u_c^+$ $\forall t \in I_i$. Evidently, in this case pair (u^*, y^*) cannot be the time-optimal solution to problem (16) because if the signal segments over the equilibrium time interval I_i are removed from signals u^* and y^* we obtain a new pair that belongs to $\mathcal{T}_{\mathbf{p}}$ and performs the required rest-to-rest transition in a time strictly less than t_f^*. Hence, $y^*(t)$ is a nonconstant function over I_i. An analogous argument runs to prove that $u^*(t)$ is a nonconstant function over O_i. □

Corollary 1 states that the time interval associated to the time-optimal control is composed of two kinds of intervals. On intervals I_i the input is saturated on the input constraints (u_c^- or u_c^+) and the output is given by a constant term plus a linear combination of the pole modes. Symmetrically, on intervals O_j the output is saturated on the constraints and the input is given by a constant term plus a linear combination of the zero modes. Hence the structure of the optimal control $u^*(t)$, denoted as *generalized bang–bang control*, is given by sequences of bang–bang functions and zero mode functions.

4. The minimum-time problem for discrete-time systems

In this section the minimum-time feedforward control problem is restated for discrete-time systems and a solution is provided using linear programming. Consider a linear discrete-time system Σ_d described by the scalar strictly proper transfer function

$$H_d(z) = \frac{b(z)}{a(z)} = \frac{b_m z^m + b_{m-1} z^{m-1} + \cdots + b_0}{z^n + a_{n-1} z^{n-1} + \cdots + a_0}. \quad (27)$$

Assume that Σ_d is stable, $a(z)$ and $b(z)$ are coprime and $H_d(1) \neq 0$. The input and output sequences are denoted by $u(k)$ and $y(k)$ respectively, $k \in \mathbb{Z}$. The behavior $\mathcal{B}_d$ of system Σ_d is the set of all input–output pairs $(u(\cdot), y(\cdot))$ satisfying the associated difference equation:

$$y(k+n) + a_{n-1} y(k+n-1) + \cdots + a_0 y(k) \\ = b_m u(k+m) + b_{m-1} u(k+m-1) + \cdots + b_0 u(k). \quad (28)$$

The set of input–output equilibrium points of Σ_d is $\mathcal{E}_d := \{(u,y) \in \mathbb{R}^2 : y = H_d(1)u\}$ and the set $\mathcal{K}_{\mathbf{p}} \subset \mathcal{B}_d$ of all rest-to-rest constrained transitions from $(0,0) \in \mathcal{E}_d$ to $(\frac{y_f}{H_d(1)}, y_f) \in \mathcal{E}_d$ is defined as follows.

Definition 3. Let be given a constraint parameter set $\mathbf{p} := \{U_c, Y_c, y_f\}$ where $U_c = [u_c^-, u_c^+]$ and $Y_c = [y_c^-, y_c^+]$ are the constraint intervals for the input and output respectively and y_f is the final output rest value for which

$$\{0, y_f\} \subset Y_c \quad \text{and} \quad \left\{0, \frac{y_f}{H_d(1)}\right\} \subset U_c.$$

Then define $\mathcal{K}_{\mathbf{p}}$ as the set of all pairs $(u(\cdot), y(\cdot)) \in \mathcal{B}_d$ for which there exists $k_f \in \mathbb{N}$ such that:

$$u(k) = 0 \quad \forall k < 0, \qquad u(k) = \frac{y_f}{H_d(1)} \quad \forall k \geq k_f, \quad (29)$$

$$u(k) \in U_c \quad k = 0, \ldots, k_f - 1, \quad (30)$$

$$y(k) = 0 \quad \forall k < 0, \qquad y(k) = y_f \quad \forall k \geq k_f, \quad (31)$$

$$y(k) \in Y_c \quad k = 0, \ldots, k_f - 1. \quad (32)$$

The following result is the discrete counterpart of Theorem 1. Its proof is analogous to that of Theorem 1 and is omitted for brevity.

Theorem 3. *Set $\mathcal{K}_{\mathbf{p}}$ is not empty if*

$$\left\{0, \frac{y_f}{H_d(1)}\right\} \subset (u_c^-, u_c^+) \quad \text{and} \quad \{0, y_f\} \subset (y_c^-, y_c^+). \quad (33)$$

The minimum-time feedforward constrained control problem for discrete-time systems consists in finding the optimal input sequence $u^*(k)$, $k = 0, 1, \ldots, k_f^* - 1$ for which the pair $(u^*(\cdot), y^*(\cdot)) \in \mathcal{K}_{\mathbf{p}}$ is a minimizer for the optimization problem:

$$k_f^* = \min_{(u(\cdot),y(\cdot)) \in \mathcal{K}_{\mathbf{p}}} K_f(u(\cdot), y(\cdot)). \quad (34)$$

$K_f(u(\cdot), y(\cdot))$, the rest-to-rest transition time associated to pair $(u(\cdot), y(\cdot))$, is defined as follows

$$K_f(u(\cdot), y(\cdot)) \\ := \min \left\{ k_1 \in \mathbb{N} : u(k) = \frac{y_f}{H_d(1)}, y(k) = y_f, \forall k \geq k_1 \right\}.$$

The key result upon which to build the solution to (34) is given by next proposition. The unit impulse response of Σ_d is denoted by $h_d(k) := \mathcal{Z}^{-1}[H_d(z)]$ and $\mathbf{1}_k$ denotes the k-dimensional vector whose components are all equal to 1.

Proposition 3. *The set $\mathcal{K}_{\mathbf{p}}$ of all rest-to-rest constrained transitions is not empty if and only if there exist $k_f \in \mathbb{N}$ and a vector $\mathbf{u} \in \mathbb{R}^{k_f}$ for which the following linear programming (LP) problem is feasible:*

$$u_c^- \cdot \mathbf{1}_{k_f} \leq \mathbf{u} \leq u_c^+ \cdot \mathbf{1}_{k_f} \quad (35)$$

$$y_c^- \cdot \mathbf{1}_{k_f} \leq \mathbf{H}\mathbf{u} \leq y_c^+ \cdot \mathbf{1}_{k_f} \quad (36)$$

$$\bar{\mathbf{H}} \begin{bmatrix} \mathbf{u} \\ \frac{y_f}{H_d(1)} \cdot \mathbf{1}_n \end{bmatrix} = y_f \cdot \mathbf{1}_n \quad (37)$$

where $\mathbf{H} \in \mathbb{R}^{k_f \times k_f}$ is defined by $\mathbf{H}_{ij} := h_d(i-j)$ and $\bar{\mathbf{H}} \in \mathbb{R}^{n \times (k_f+n)}$ by $\bar{\mathbf{H}}_{ij} := h_d(i + k_f - j)$.

Proof (*Sufficiency*). Assume that there exist $k_f \in \mathbb{N}$ and a vector $\mathbf{u} = [u_0, u_1, \ldots, u_{k_f-1}]^T$ for which Eqs. (35)–(37) are satisfied. Define the input sequence

$$u(k) = \begin{cases} 0 & \text{if } k < 0 \\ u_k & \text{if } 0 \leq k < k_f \\ \frac{y_f}{H_d(1)} & \text{if } k \geq k_f, \end{cases} \quad (38)$$

which satisfies Properties (29) and (30) of Definition 3. The output is given by $y(k) = \sum_{i=0}^{\infty} u(k-i) h_d(i)$. Setting $\mathbf{y} = [y_0, y_1, \ldots, y_{k_f-1}]^T \in \mathbb{R}^{k_f}$ and $\bar{\mathbf{y}} = [\bar{y}_0, \bar{y}_1, \ldots, \bar{y}_{n-1}]^T \in \mathbb{R}^n$, according to $\mathbf{y} = \mathbf{H}\mathbf{u}$, $\bar{\mathbf{y}} = \bar{\mathbf{H}} \begin{bmatrix} \mathbf{u} \\ \frac{y_f}{H_d(1)} \cdot \mathbf{1}_n \end{bmatrix}$, it follows that

$$y(i) = \mathbf{y}_i, \quad i = 0, 1, \ldots, k_f - 1$$
$$y(k_f + i) = \bar{\mathbf{y}}_i, \quad i = 0, 1, \ldots, n - 1$$

and, by (36), sequence $y(k)$ satisfies the constraint (32) of Definition 3. It remains to show that $y(i) = y_f, \forall i \geq k_f + n$. To prove this, set $k = k_f$ in difference equation (28), noting that $H_d(1) = \frac{b_m + b_{m-1} + \cdots + b_0}{1 + a_{n-1} + \cdots + a_0}$ it follows that $y(k_f + n) = y_f$. By iteration we have $y(k) = y_f, \forall k > k_f + n$. Indeed condition (37) guarantees that at $k = k_f$ the system has reached the equilibrium.

(*Necessity*). Assume that the set $\mathcal{K}_{\mathbf{p}}$ is nonempty, therefore there exists $k_f \in \mathbb{N}$ and a pair $(u(k), y(k))$ which satisfies conditions (29)–(32). Define $\mathbf{u} = [u(0), u(1), \ldots, u(k_f - 1)]^T$, then (35) follows from (30) and inequality (36) follows from (32) and the fact that $[y(0), y(1), \ldots, y(k_f - 1)]^T = \mathbf{H}\mathbf{u}$. Finally (37) follows from (31) and the fact that

$$\begin{bmatrix} y(k_f) \\ y(k_f+1) \\ \vdots \\ y(k_f+n-1) \end{bmatrix} = \bar{\mathbf{H}} \begin{bmatrix} \mathbf{u} \\ \frac{y_f}{H_d(1)} \cdot \mathbf{1}_n \end{bmatrix}. \quad \square$$

By virtue of Proposition 3, the minimum number of steps k_f^* and an associated optimal feedforward input $u^*(k)$, $k = 0, 1, \ldots k_f^* - 1$ can be determined by means of a sequence of LP feasibility tests (the problem defined at (35)–(37)) through the simple bisection algorithm reported below. In this algorithm $LPP(\mathbf{p}, k_f, \mathbf{u})$ denotes a linear programming procedure that solves problem (35)–(37): if the problem is feasible it returns a Boolean true value along with a solution $\mathbf{u} \in \mathbb{R}^{k_f}$. This solution vector $\mathbf{u}$ defines a corresponding input sequence according to

$$[u(0), u(1), \ldots, u(k_f - 1)]^T = \mathbf{u}. \quad (39)$$

Minimum-time feedforward constrained regulation algorithm

Input: $H_d(z)$ and $\mathbf{p} = \{U_c, Y_c, y_f\}$
Output: k_f^* and $\mathbf{u}^*$ that corresponds to an optimal control sequence $u^*(k)$ according to (39).

$k_f \longleftarrow 1$
$l \longleftarrow 0$
while $\sim LPP(\mathbf{p}, k_f, \mathbf{u})$ **do**
  $l \longleftarrow k_f$
  $k_f \longleftarrow 2k_f$
end while
$h \longleftarrow k_f$
while $h - l > 1$ **do**
  $k_f \longleftarrow \lfloor \frac{h+l}{2} \rfloor$
  if $\sim LPP(\mathbf{p}, k_f, \mathbf{u})$ **then** $l \longleftarrow k_f$
  else $h \longleftarrow k_f$
  end if
end while
$k_f^* \longleftarrow h$
$\mathbf{u}^* \longleftarrow \mathbf{u}$

Remark 2. Differently from the continuous-time case, the discrete-time optimal solution $u^*(k)$ is not unique (see Desoer and Wing (1961)).

5. An approximated solution to the continuous-time problem using discretization

The procedure developed in Section 4 allows to find the optimal minimum-time constrained transition for discrete-time systems. This section shows that it can be used to find an approximated solution to the continuous-time problem. Given the continuous-time system Σ (1), an approximation to the optimal generalized bang–bang control $u^*(t)$ can be found as follows:

- Choose a sampling period T and determine the discretized system using a zero-order equivalence, by relation $H_T(z) = (1-z^{-1})\mathcal{Z}\left[\frac{H(s)}{s}\right]$, where $\mathcal{Z}[P(s)] = \sum_{i=0}^{+\infty} p(kT)z^{-i}$ and $p(t) = \mathcal{L}^{-1}[P(s)]$ is the impulse response of a system with transfer function $P(s)$.
- Find a minimum-time input sequence $u_T^*(k)$, using the algorithm described in Section 4.
- An approximated continuous-time solution is given by

$$u_T^*\left(\left\lfloor \frac{t}{T} \right\rfloor\right). \tag{40}$$

The following result shows that solution (40) can be made arbitrarily close to the optimal one, by choosing a sufficiently small sampling time T.

Theorem 4. *Assume that inclusions (8) of Theorem 1 are satisfied. Let t_f^* be the optimal time as defined in (16) and let (u^*, y^*) be the associated optimal pair. Let $k_f^*(T)$ be the minimum number of steps defined by (34) relative to system $H_T(z)$ and let (u_T^*, y_T^*) be the associated optimal sequence pair. Then the following limits hold*

$$\lim_{T\to 0} k_f^*(T)T = t_f^*, \tag{41}$$

$$\lim_{T\to 0} u_T^*\left(\left\lfloor \frac{t}{T} \right\rfloor\right) = u^*(t), \quad a.e.$$
$$\lim_{T\to 0} y_T^*\left(\left\lfloor \frac{t}{T} \right\rfloor\right) = y^*(t), \quad a.e. \tag{42}$$

Proof. Limit (41) is equivalent to the following two inequalities

$$\liminf_{T\to 0} k_f^*(T)T \geq t_f^* \tag{43}$$

$$\limsup_{T\to 0} k_f^*(T)T \leq t_f^*. \tag{44}$$

First, to prove (43), we assume by contradiction that there exists $\sigma > 0$ for which

$$t_f^* - \liminf_{T\to 0} k_f^*(T)T = \sigma \tag{45}$$

and show that, as a consequence, there exists a continuous-time input–output pair that performs the constrained rest-to-rest transition in a time less than t_f^*. By assumption (45) there exists an infinite sequence of decreasing sampling times $T_i > 0$, $i \in \mathbb{N}$, such that $\lim_{i\to\infty} T_i = 0$ and the following two properties are verified

$$\lim_{i\to\infty} t_f^* - k_f^*(T_i)T_i = \sigma, \tag{46}$$

$$t_f^* - k_f^*(T_i)T_i \in \left[\frac{3}{4}\sigma, \frac{5}{4}\sigma\right], \quad \forall i \in \mathbb{N}. \tag{47}$$

Set $c = \left(|h(0^+)| + \|Dh\|_1\right)\max\{|u_c^-|, |u_c^+|\}$, $y_M = \min\{|y_c^-|, |y_c^+|\}$ and define the continuous-time input $u_i(t) = u_{T_i}^*(\lfloor t/T_i \rfloor)\frac{y_M - cT_i}{y_M}$. If the corresponding output is given by $y_i(t) = \int_0^t h(t-v)u_i(v)\mathrm{d}v$ then it satisfies the property $y_i(kT_i) = y_{T_i}^*(k)\frac{y_M - cT_i}{y_M} \in [y_c^-, y_c^+]$, $\forall k \in \mathbb{Z}$. By Lemma 4 (see Appendix B), $\forall t \geq 0$, $\forall i \in \mathbb{N}$:

$$y_i(t) \leq y_c^+ \frac{y_M - cT_i}{y_M} + cT_i \leq \frac{y_c^+(y_M - cT_i) + cT_i y_M}{y_M}$$
$$\leq y_c^+ + cT_i \frac{y_M - y_c^+}{y_M} \leq y_c^+,$$

and analogously $y_i(t) \geq y_c^-$. Therefore the pair (u_i, y_i) satisfies the input–output constraints and reaches final rest conditions because $\forall t \geq T_i k_f^*(T_i)$, $u_i(t) = y_f \frac{y_M - cT_i}{H(0)y_M}$ and, by Lemma 3 (see Appendix B) $y_i(t) = y_f \frac{y_M - cT_i}{y_M}$. However, $(u_i, y_i) \notin \mathcal{T}_{\mathbf{p}}$ so that to enforce the required final rest conditions, in time interval $[T_i k_f^*(T_i), T_i k_f^*(T_i) + \sigma/2]$ we add a correcting term to the input u_i as follows. Apply Lemma 5 (Appendix B) to find a correcting pair $(\tilde{u}_i, \tilde{y}_i)$ such that $\tilde{u}_i(t) = 0$ if $t < T_i k_f^*(T_i)$, $\tilde{u}_i(t) = \frac{y_f cT_i}{H(0)y_M}$, if $t > T_i k_f^*(T_i) + \sigma/2$ and

$$\tilde{y}_i(T_i k_f^*(T_i) + \sigma/2) = y_f \frac{cT_i}{y_M}$$
$$D\tilde{y}_i(T_i k_f^*(T_i) + \sigma/2) = 0$$
$$\vdots$$
$$D^{n-1}\tilde{y}_i(T_i k_f^*(T_i) + \sigma/2) = 0.$$

Then define the pair $(\hat{u}_i, \hat{y}_i) = (u_i + \tilde{u}_i, y_i + \tilde{y}_i)$ for which $\hat{y}_i(t) = y_f$, $\forall t \geq T_i k_f^*(T_i) + \sigma/2$. Moreover in the interval $T_i k_f^*(T_i) < t < T_i k_f^*(T_i) + \sigma/2$

$$\hat{u}_i(t) \leq y_f H(0)^{-1}\left(\frac{y_M - cT_i}{y_M} + M_u \frac{cT_i}{y_M}\right),$$

$$\hat{y}_i(t) \leq y_f \frac{y_M - cT_i}{y_M} + M_y \frac{cT_i}{y_M},$$

where M_u and M_y are constants (note that the length of the correction is given by $\frac{\sigma}{2}$ and is fixed for all i). By choosing a sufficiently large i (and, hence, a sufficiently small T_i), the input and output constraints can always be satisfied. Therefore, there exists a continuous-time input–output pair that performs the constrained rest-to-rest transition in time $T_i k_f^*(T_i) + \frac{\sigma}{2}$. Hence, by (47), $T_i k_f^*(T_i) + \frac{\sigma}{2} \leq t_f^* - \frac{\sigma}{4} < t_f^*$. This last inequality contradicts the optimality of t_f^* so that proof of (43) is completed. In order to prove limit (44), assume by contradiction that there exists $\sigma > 0$ for which

$$\limsup_{T\to 0} k_f^*(T)T - t_f^* = \sigma. \tag{48}$$

Hence, there exists an infinite sequence of decreasing sampling times $T_i > 0$, $i \in \mathbb{N}$, such that $\lim_{i\to\infty} T_i = 0$ and

$$k_f^*(T_i)T_i - t_f^* \in \left[\frac{3}{4}\sigma, \frac{5}{4}\sigma\right], \quad \forall i \in \mathbb{N}. \tag{49}$$

The sequence $h_T(k)$ represents the impulse response of the discretized system obtained through a zero-order hold with sampling time T and is given by $h_T(k) = \int_0^T h(kT - t)\mathrm{d}t$. Define the input sequence $u_T(k) = u^*(Tk)$, where $u^*(t)$ is the time-optimal continuous-time control. The corresponding output sequence is given by $y_T(k) = \sum_{i=0}^{\infty} h_T(k-i)u_T(k)$. Consider the difference between the sampled continuous-time optimal output $y^*(Tk)$ and the discrete-time system output $y_T(k)$:

$$y^*(Tk) - y_T(k) = \int_0^{Tk} h(Tk-t)u^*(t)\mathrm{d}t - \sum_{i=0}^{k-1}\left[\int_0^T h(Tk-Ti-t)\mathrm{d}t\right]u_T(i)$$
$$= \int_0^{Tk} h(Tk-t)u^*(t)\mathrm{d}t - \int_0^{Tk} h(Tk-t)u^*\left(\left\lfloor\frac{t}{T}\right\rfloor T\right)\mathrm{d}t$$
$$= \int_0^{Tk} h(Tk-t)\left(u^*(t) - u^*\left(\left\lfloor\frac{t}{T}\right\rfloor T\right)\right)\mathrm{d}t.$$

Since $u^*(t)$ is continuous almost everywhere, it follows that $\lim_{T\to 0}\left(u^*(t) - u^*(\lfloor\frac{t}{T}\rfloor T)\right) = 0$, a.e. Hence, by Lebesgue dominated convergence theorem

$$\lim_{T\to 0} y^*(Tk) - y_T(k) = 0. \tag{50}$$

Therefore, for any $\epsilon > 0$, there exists $T_\epsilon > 0$ sufficiently small such that $|y^*(T_\epsilon k) - y_{T_\epsilon}(k)| < \epsilon, \forall k \in \mathbb{Z}$. Set $k_T = \lceil t_f^*/T\rceil$, $y_M = \min\{|y_c^-|, |y_c^+|\}$ and consider the following input–output sequences

$$\bar{u}_{T_\epsilon}(k) = u_{T_\epsilon}(k)\frac{y_M - \epsilon}{y_M}, \qquad \bar{y}_{T_\epsilon}(k) = y_{T_\epsilon}(k)\frac{y_M - \epsilon}{y_M},$$

for which $\bar{u}_{T_\epsilon}(k) \in [u_c^-, u_c^+]$, $\bar{y}_{T_\epsilon}(k) \in [y_c^-, y_c^+]$, $\forall k > 0$, i.e. the pair $(\bar{u}_{T_\epsilon}, \bar{y}_{T_\epsilon})$ satisfies the input–output constraints. The required final rest condition is not satisfied and it is necessary to perform a correction on $(\bar{u}_{T_\epsilon}, \bar{y}_{T_\epsilon})$, following the same reasoning done in the first part of this proof. A correcting discrete-time input is added in the interval $k_{T_\epsilon} \le k < k_{T_\epsilon} + nl$ to enforce the final equilibrium condition, where $l = \lfloor\sigma/(4T_\epsilon n)\rfloor$. The correcting input sequence will be constant every l consecutive steps. Consider $h_{T_\epsilon l}(k) = \int_0^{T_\epsilon l} h(kT_\epsilon l - t)\mathrm{d}t$, and define matrix $\mathbf{W}(T_\epsilon l) \in \mathbb{R}^{n\times n}$ according to $\mathbf{W}(T_\epsilon l)_{ij} = h_{T_\epsilon l}(n+j-i)$. Given a vector $\mathbf{a} = [a_0, a_1, \ldots, a_{n-1}] \in \mathbb{R}^n$, if

$$\tilde{u}(k) = \begin{cases} 0 & \text{if } k < k_{T_\epsilon} \text{ or } k \ge k_{T_\epsilon} + nl \\ a_i & \text{if } \left\{k_{T_\epsilon} + il \le k < k_{T_\epsilon} + (i+1)l, \quad i = 0, \ldots, n-1,\right. \end{cases}$$

define $\mathbf{b} = [b_0, b_1, \ldots, b_{n-1}] = \mathbf{W}(T_\epsilon l)\mathbf{a}$, then $\tilde{y}(t) = \int_{-\infty}^t \tilde{u}(\tau)h_{T_\epsilon}(t-\tau)\mathrm{d}\tau$ satisfies $\tilde{y}\left(k_{T_\epsilon} + (n+i)l\right) = b_i, i = 0, \ldots, n-1$. Define the correction input $\tilde{u}_{T_\epsilon}(k)$ as follows

$$\tilde{u}_{T_\epsilon}(k) = 0 \quad \text{if } k < k_{T_\epsilon},$$
$$\tilde{u}_{T_\epsilon}(k) = \mathbf{W}(T_\epsilon l)^{-1}\cdot\left(\begin{bmatrix} y_f - \bar{y}_{T_\epsilon}(k_{T_\epsilon}) \\ y_f - \bar{y}_{T_\epsilon}(k_{T_\epsilon} + l) \\ \cdots \\ y_f - \bar{y}_{T_\epsilon}(k_{T_\epsilon} + (n-1)l)\end{bmatrix} - \frac{\epsilon}{y_M}\frac{y_f}{H(0)}\begin{bmatrix} h_{T_\epsilon l}(0) \\ h_{T_\epsilon l}(0) + h_{T_\epsilon l}(1) \\ \cdots \\ \sum_{i=0}^{n-1} h_{T_\epsilon l}(i)\end{bmatrix}\right) \quad \text{if } k_{T_\epsilon} \le k < k_{T_\epsilon} + ln,$$
$$\tilde{u}_{T_\epsilon}(k) = \frac{\epsilon}{y_M}\frac{y_f}{H(0)} \quad \text{if } k \ge k_{T_\epsilon} + ln.$$

Finally, define the corrected input–output pair by $(\hat{u}_{T_\epsilon}, \hat{y}_{T_\epsilon}) = (\bar{u}_{T_\epsilon} + \tilde{u}_{T_\epsilon}, \bar{y}_{T_\epsilon} + \tilde{y}_{T_\epsilon})$. Then, $\hat{y}_{T_\epsilon}(k) = \bar{y}_{T_\epsilon}(k)$ if $k < k_{T_\epsilon}$, moreover $\hat{y}_{T_\epsilon}(k_{T_\epsilon} + nl + kl) = y_f$, $k = 0, \ldots, n-1$ and, by Lemma 3, $\hat{y}_{T_\epsilon}(k) = y_f, \forall k \ge k_{T_\epsilon} + nl$. It remains to show that the input and output constraints are satisfied for $k_{T_\epsilon} \le k < k_{T_\epsilon} + nl$. Consider that $\forall k \ge k_{T_\epsilon}$, $|y_f - \bar{y}_{T_\epsilon}(k)| \le |y_f - y_{T_\epsilon}(k)| + |\frac{\epsilon}{y_M}y_{T_\epsilon}(k)| \le \epsilon(2 + \frac{\epsilon}{y_M})$, and define $B_\epsilon = \epsilon\sqrt{n}\|\mathbf{W}(T_\epsilon l)^{-1}\|\left(2 + \frac{\epsilon}{y_M} + \frac{n\|h\|_1}{H(0)}\right)$. Therefore, for $k_{T_\epsilon} \le k < k_{T_\epsilon} + nl$

$$y_f\frac{y_M - \epsilon}{H(0)y_M} - B_\epsilon \le \hat{u}_{T_\epsilon}(k) \le y_f\frac{y_M - \epsilon}{H(0)y_M} + B_\epsilon,$$
$$y_f\frac{y_M - \epsilon}{y_M} - \|h\|_1 B_\epsilon - \epsilon \le \hat{y}_{T_\epsilon}(k) \le y_f\frac{y_M - \epsilon}{y_M} + \|h\|_1 B_\epsilon + \epsilon.$$

Term $\|\mathbf{W}(T_\epsilon l)^{-1}\|$ is bounded for any $T_\epsilon > 0$ because quantity $T_\epsilon l$ is included in a compact interval according to $T_\epsilon l = T_\epsilon\lfloor\frac{\sigma}{2T_\epsilon n}\rfloor \in [\frac{\sigma}{2n} - T_\epsilon, \frac{\sigma}{2n} + T_\epsilon]$ and $\mathbf{W}(T_\epsilon l)$ is a continuous function of its argument. This means that $\lim_{T_\epsilon\to 0}\|\mathbf{W}^{-1}(T_\epsilon l)\|\frac{\epsilon}{y_M} = 0$. Choose $\bar{\epsilon} > 0$ (and consequently $T_{\bar{\epsilon}}$) sufficiently small such that pair $(\hat{u}_{T_\epsilon}, \hat{y}_{T_\epsilon})$ satisfies the input and output constraints and $\lceil t_f^*/T_{\bar{\epsilon}}\rceil T_{\bar{\epsilon}} - t_f^* < \sigma/4$, i.e., $k_{T_{\bar{\epsilon}}}T_{\bar{\epsilon}} - t_f^* < \sigma/4$. Considering the introduced sequence $\{T_i\}$ of decreasing sampling times, there exists $r \in \mathbb{N}$ such that $T_r \le T_{\bar{\epsilon}}$ and

$$k_{T_r}T_r - t_f^* < \sigma/4, \tag{51}$$

$$k_f^*(T_r)T_r - t_f^* \in \left[\frac{3}{4}\sigma, \frac{5}{4}\sigma\right]. \tag{52}$$

Pair $(\hat{u}_{T_r}, \hat{y}_{T_r})$ satisfies the input and output constraints and performs the required rest-to-rest transition in $k_{T_r} + nl$ steps. Taking into account that $nl \le \frac{\sigma}{4T_r}$, from (51) $\left(k_{T_r} + nl\right)T_r - t_f^* < \sigma/2$, and from (52) $k_f^*(T_r)T_r - t_f^* \ge (3/4)\sigma$. Therefore, $k_f^*(T_r) > k_{T_r} + nl$ and this violates the optimality of $k_f^*(T_r)$. This completes the proof of (44) and therefore (41) holds.

Let T_i be a sequence of decreasing sampling times such that $\lim_{i\to\infty} T_i = 0$. Hence, limit (41) holds, i.e., $\lim_{i\to\infty} k_f^*(T_i)T_i = t_f^*$. Consider the sequence pairs $(u_{T_i}^*(k), y_{T_i}^*(k))$ and apply the procedure devised in the first part of this proof to obtain continuous-time pairs $(\hat{u}_i(t), \hat{y}_i(t))$ for which, when $i \to \infty$, the transition time $T_f(\hat{u}_i, \hat{y}_i)$ converges to t_f^*. By Proposition 8 in Appendix A, the sequence $(\hat{u}_i(t), \hat{y}_i(t))$ converges a.e. to the unique optimal pair $(u^*(t), y^*(t))$ as $i \to \infty$. Since $\lim_{i\to\infty}\|u_{T_i}^*(\lfloor\frac{t}{T_i}\rfloor) - \hat{u}_i(t)\|_\infty = 0$ and $\lim_{i\to\infty}\|y_{T_i}^*(\lfloor\frac{t}{T_i}\rfloor) - \hat{y}_i(t)\|_\infty = 0$, the pairs $(u_{T_i}^*(\lfloor\frac{t}{T_i}\rfloor), y_{T_i}^*(\lfloor\frac{t}{T_i}\rfloor))$ converge a.e. to $(u^*(t), y^*(t))$ when $i \to \infty$ and therefore (42) holds. □

6. Examples

Example 1. Consider a continuous-time system described by transfer function $H_1(s) = \frac{10(s+2)}{(s+1)^2+9}$. We desire a rest-to-rest transition from $y = 0$ to $y = 3(=y_f)$ to be completed in minimum-time with amplitude input constraints defined by $U_c = [u_c^-, u_c^+] = [-1.8, 1.8]$. In a first case no output constraints are considered, i.e. $Y_c = (-\infty, \infty)$, and in a second case we impose $Y_c = [y_c^-, y_c^+] = [-0.1, 3.1]$. This corresponds to regulation constraints given by a maximum 3.3% overshooting and 3.3% undershooting. The system static gain is $H_1(0) = 2$ and conditions (8) of Theorem 1 are satisfied: $\{0, 1.5\} \subset (-1.8, 1.8)$, $\{0, 3\} \subset (-\infty, \infty)$ and $\{0, 1.5\} \subset (-1.8, 1.8)$, $\{0, 3\} \subset (-0.1, 3.1)$. Hence the minimum-time feedforward constrained regulation problem has solution in both cases. The optimal control $u^*(t)$ is computed by applying the discretization procedure of Section 5 with sampling period $T = 0.002$ s.

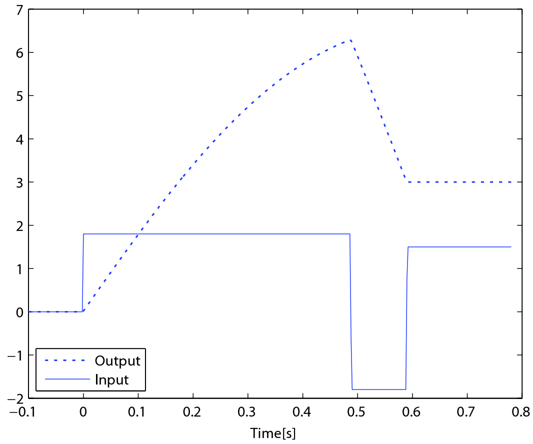

Fig. 1. Example 1, bang–bang control.

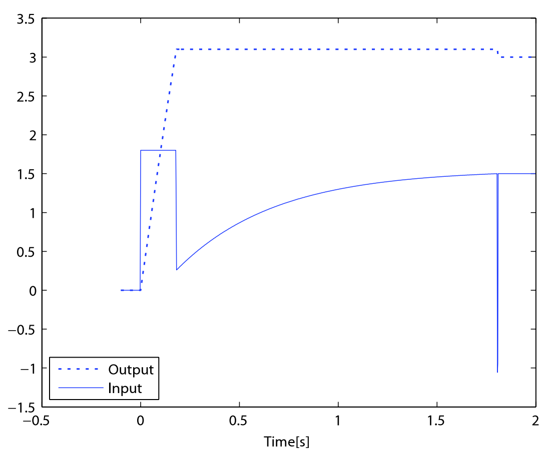

Fig. 2. Example 1, generalized bang–bang control.

The results are exposed in Figs. 1 and 2. Both figures plot the pair $(u^*(\cdot), y^*(\cdot))$ over the optimal transition interval. Fig. 1 shows that $u^*(t)$ is the well-known bang–bang control that permits to obtain the minimum-time $t_f^* = 0.6805$ s at the price of a large overshooting (more than 100% of the final rest value). In the second case, due to the imposed output constraints, the overshooting is almost completely removed (see Fig. 2) and the resulting optimal feedforward control $u^*(t)$ is composed of a bang–bang function followed by a zero dynamics mode (in which the output saturates the constraint) and a final short bang–bang spike. The associated minimum-time is $t_f^* = 1.898$ s.

Example 2. This last example considers a system with transfer function $H_2(s) = \frac{10(3.5-s)(s^2+25)}{(s+2)(s+3)(s+4)(s+5)}$. The required rest-to-rest transition is from $y = 0$ to $y = 3$. The constraint intervals are $U_c = [-2, +2]$ and $Y_c = [-0.1, +3.1]$. Again conditions (8) of Theorem 1 are satisfied. The optimal $u^*(t)$ and the corresponding $y^*(t)$ are plotted in Fig. 3. The input is composed of a bang–bang spike, a zero mode function and a bang–bang function. The achieved minimum-time is $t_f^* = 1.382$ s. The sampling time used for the computation is $T = 0.002$ s. It is worth noting the intricate behavior of the optimal $y^*(t)$: after a relatively long time plateau the output increases till to a local maximum, then decreases till to a local minimum and finally reaches the desired rest position. The surprising details of the optimal input–output pair are due to the intrinsic difficulty in regulating a system with both an unstable zero and a couple of purely imaginary zeros.

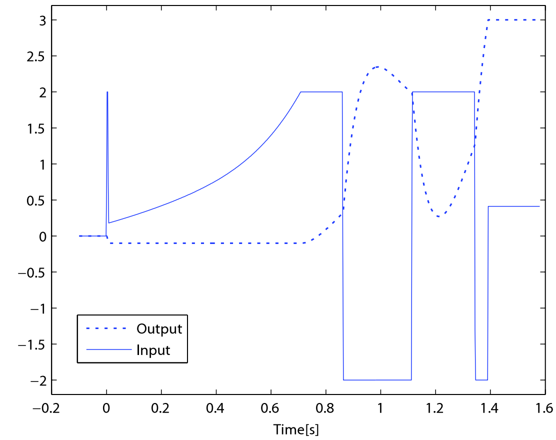

Fig. 3. Example 2, generalized bang–bang control.

7. Conclusions

This paper has posed a new minimum-time feedforward regulation problem with input and output amplitude constraints. The provided solution leads to a generalization of the classic bang–bang control that can be determined by means of a discretization procedure based on linear programming feasibility tests. A novelty of the proposed approach to constrained regulation is the ability to deal with both (i) arbitrarily stringent constraints on input and output and (ii) nonminimum-phase plants with purely imaginary zeros. This appears a significant improvement over the inversion-based approach to feedforward constrained regulation (Piazzi & Visioli, 2001, 2005).

An interesting extension of the proposed approach would be the MIMO (multi-input multi-output) case. Conceptually, the MIMO solution should still exhibit a generalized bang–bang structure (i.e. almost at all times at least one of the inputs or one of the outputs saturates on the constraint). However possible degeneracies may emerge in the non-square case (when the number of inputs and outputs are different). This will be investigated in future research. The generalized bang–bang control seems a technique that can be applied to a broad range of applications. First results in process control and mechatronics have recently appeared (Consolini, Gerelli, Guarino Lo Bianco, & Piazzi, 2009; Consolini, Piazzi, & Visioli, 2007).

Appendix A. Existence and uniqueness of the solution to the minimum-time feedforward constrained regulation problem

First we recall a result from Polderman and Willems (1998) regarding the closedness of the system behavior set $\mathcal{B}$.

Proposition 4. *If* $(u_i, y_i) \in \mathcal{B}$ $i \in \mathbb{N}$ *is a sequence converging to* $(\tilde{u}, \tilde{y})$ *in the sense of* L_1^{loc}, *then* $(\tilde{u}, \tilde{y}) \in \mathcal{B}$.

The following definition introduces a subset of $\mathcal{T}_\mathbf{p}$ that represents the input–output pairs that perform the constrained rest-to-rest transition with a transition time less or equal than M.

Definition 4. Given a real number $M > 0$, the set of constrained rest-to-rest transitions with transition time bounded by M is given by $\mathcal{T}_\mathbf{p}^M = \{(u, y) \in \mathcal{T}_\mathbf{p} : T_f(u, y) \le M\}$.

The following proposition shows that set $\mathcal{T}_\mathbf{p}^M$ is compact in the sense of L_1.

Proposition 5. *Given any sequence of input–output pairs* $(u_i, y_i) \in \mathcal{T}_\mathbf{p}^M$, *there exists a subsequence* (u_{l_i}, y_{l_i}) *and a pair* $(u, y) \in \mathcal{T}_\mathbf{p}^M$, *such that*

$$\lim_{i\to\infty} \int_0^M \left(|u - u_{l_i}| + |y - y_{l_i}|\right) dt = 0.$$

Proof. Define the functional

$$\tilde{T}_f(u,y) = \inf\left\{t_f \geq 0 : \forall (a,b) \subset (t_f,+\infty) : \int_a^b \left|u - \frac{y_f}{H(0)}\right| \mathrm{d}t = 0, \int_a^b |y-y_f|\mathrm{d}t = 0\right\}.$$

First of all we prove that $\tilde{T}_f$ is a lower semicontinuous functional. This is equivalent to checking that

$$\tilde{T}_f^{-1}((c,+\infty)) = \left\{(u,y)|\exists (a,b) \subset (c,+\infty), \epsilon > 0 : \int_a^b \left|u - \frac{y_f}{H(0)}\right| \mathrm{d}t = \epsilon \text{ or } \int_a^b |y-y_f|\mathrm{d}t = \epsilon\right\}$$

is an open set (see Theorem 7.1.1 of Kurdila and Zabarankin (2005)). Choose $(u_1,y_1) \in \tilde{T}_f^{-1}((c,+\infty))$, and consider the open ball centered in (u_1,y_1):

$$B_{\epsilon/2} = \left\{(u,y) \in L_1^{\text{loc}} \times L_1^{\text{loc}} | \|(u-u_1, y-y_1)\|_1 < \frac{\epsilon}{2}\right\}.$$

Assume for instance that $\int_a^b |u_1 - \frac{y_f}{H(0)}|\mathrm{d}t = \epsilon$, the case in which $\int_a^b |y_1 - y_f|\mathrm{d}t = \epsilon$ is analogous. For $(u,y) \in B_{\epsilon/2}$

$$\int_a^b \left|u - \frac{y_f}{H(0)}\right| \mathrm{d}t \geq \int_a^b \left|u_1 - \frac{y_f}{H(0)}\right| \mathrm{d}t - \int_a^b |u-u_1|\mathrm{d}t \geq \frac{\epsilon}{2}$$

then $\tilde{T}_f(u,y) \geq a > c$ and $B_{\epsilon/2} \in \tilde{T}_f^{-1}((c,+\infty))$ and $\tilde{T}_f^{-1}((c,+\infty))$ is open and the complementary set $\tilde{T}_f^{-1}(-\infty,c]$ is closed. Consider the set $\tilde{\mathcal{T}}_{\mathbf{p}}^M = \{(u,y) \in \tilde{\mathcal{T}}_{\mathbf{p}} : \tilde{T}_f(u,y) \leq M\}$, this can be written as $\tilde{\mathcal{T}}_{\mathbf{p}}^M = E \cap \mathcal{B} \cap \tilde{T}_f^{-1}(-\infty, M]$, where $E = \{(u,y)|u(t) \in U_c, y(t) \in Y_c, \forall t \in [0,M]\}$ and $\mathcal{B}$ is the behavior set. By a trivial continuous linear affine transformation it is possible to map the set E on the unit ball of $L_\infty \times L_\infty$, then by Alaoglu's Theorem, set E is weakly* compact (see Theorem 7.3.2 of Kurdila and Zabarankin (2005)). This means that for every sequence of functions $(u_l, y_l) \in E$, there exists a pair $(u,y) \in E$ and a subsequence l_i such that $\forall f, g \in L_1$ it is $\lim_{i\to\infty} \int_0^M ((u-u_i)f + (y-y_i)g)\,\mathrm{d}t = 0$, in particular it follows that $\lim_{i\to\infty} \int_0^M (|u-u_i| + |y-y_i|)\,\mathrm{d}t = 0$, therefore E is a compact set in the (strong) topology of L_1. Moreover $\tilde{\mathcal{T}}_{\mathbf{p}}^M$ is compact because is the intersection of the compact set E with the closed sets $\mathcal{B}$ and $\tilde{T}_f^{-1}(-\infty,M]$. Since $\tilde{\mathcal{T}}_{\mathbf{p}}^M$ is compact and $\mathcal{T}_{\mathbf{p}}^M \subset \tilde{\mathcal{T}}_{\mathbf{p}}^M$, there exists a pair $(u,y) \in \tilde{\mathcal{T}}_{\mathbf{p}}^M$ and a subsequence l_i such that $\lim_{i\to\infty} \int_0^M \left(|u-u_{l_i}| + |y-y_{l_i}|\right)\mathrm{d}t = 0$. Finally apply to the pair (u,y) the following flattening operator $\Pi : \tilde{\mathcal{T}}_{\mathbf{p}}^M \to \mathcal{T}_{\mathbf{p}}^M, \Pi(u,y) = (u_2,y_2)$, where $u_2(t) = u(t), \forall t \leq T_f(u,y)$, $u_2(t) = \frac{y_f}{H(0)}, \forall t > T_f(u,y)$ and analogously $y_2(t) = y(t), \forall t \leq T_f(u,y), y_2(t) = y_f, \forall t > T_f(u,y)$. Since $(u_2,y_2) \in \mathcal{T}_{\mathbf{p}}^M$ and $\lim_{i\to\infty}\int_0^M \left(|u_2-u_{l_i}| + |y_2-y_{l_i}|\right)\mathrm{d}t = 0$, the proposition is proved. □

Proposition 6. *There exists an optimal pair $(u^*,y^*) \in \mathcal{T}_{\mathbf{p}}$ such that $T_f(u^*,y^*) = t_f^*$.*

Proof. The generalized Weierstrass theorem (see 7.3.1 of Kurdila and Zabarankin (2005)) implies that $t_f^* = \inf_{\tilde{\mathcal{T}}_{\mathbf{p}}^M} \tilde{T}_f(u,y)$, where $\tilde{\mathcal{T}}_{\mathbf{p}}^M$ is a compact set and $\tilde{T}_f$ is a lower semicontinuous function as shown in the proof of Proposition 5. Let $(u,y) \in \tilde{T}_{\mathbf{p}}^M$ be the corresponding optimal pair and apply the flattening operator Π defined in the same proof, setting $(u^*,y^*) = \Pi(u,y)$, then $\forall (u,y) \in \mathcal{T}_{\mathbf{p}} T_f(u^*,y^*) = \tilde{T}_f(u^*,y^*) \leq \tilde{T}_f(u,y) \leq T_f(u,y)$, therefore (u^*,y^*) is an optimal pair. □

Consider the following notation (see Polderman and Willems (1998, page 35) for the multiple integral of a function u. Define $(\int^{(0)} u)(t) = u(t)$ and, $\forall i > 0, i \in \mathbb{N}, (\int^{(i)} u)(t) = \int_0^t (\int^{(i-1)} u)(v)\mathrm{d}v$.

Lemma 2. *Let be given a function $u(t) : \mathbb{R} \to \mathbb{R}$ and real numbers $a < b < c$, then*

(a) $\int_a^b |u(t)|\mathrm{d}t = 0$ if and only if there exist real constants $c_0, c_1, \ldots c_{n-1}$ such that $\forall t \in [a,b]$, $(\int^{(n)} u)(t) = c_0 + c_1 t + c_2 t^2 + \cdots c_{n-1}t^{n-1}$,

(b) if $\int_b^c |u(t)|\mathrm{d}t = 0$ and $\forall t \in [a,b]$, $(\int^{(n)} u)(t) = p(t)$, where $p(t)$ is a polynomial of degree $n-1$, then $\forall t \in [b,c] (\int^{(n)} u)(t) = p(t)$.

The proof is omitted for brevity.

Proposition 7. *The optimal pair (u^*,y^*) is essentially unique, i. e. if $(u,y) \in \mathcal{T}_{\mathbf{p}}$ and $T_f(u,y) = t_f^*$ then*

$$\int_0^{+\infty} \left(|u(t) - u^*(t)| + |y(t) - y^*(t)|\right)\mathrm{d}t = 0.$$

Proof. Let (u,y) be an input–output pair such that $T_f(u,y) = t_f^*$, then all convex linear combinations of the form $(u_\lambda, y_\lambda) = (1-\lambda)(u,y) + \lambda(u^*,y^*)$, satisfy $T_f(u_\lambda,y_\lambda) = t_f^*$. As a consequence of Theorem 2, $\int_0^{t_f^*} \min\left\{d\left(u_\lambda(t), \{u_c^-, u_c^+\}\right), d\left(y_\lambda(t), \{y_c^-, y_c^+\}\right)\right\}\mathrm{d}t = 0$, this implies that

$$\int_0^{+\infty} \min\{|u-u^*|, |y-y^*|\}\mathrm{d}t = 0. \tag{53}$$

The pair $(\bar{u},\bar{y}) = (u-u^*, y-y^*)$ is a weak solution of (2) and satisfies a.e.

$$B[\bar{u}](t) = A[\bar{y}](t) + p(t), \tag{54}$$

where $B[\bar{u}] = \sum_{i=0}^m b_i(\int^{(n-i+1)} \bar{u})$, $A[\bar{y}] = \sum_{i=0}^n a_i(\int^{(n-i+1)} \bar{y})$ and $p(t)$ is a suitable polynomial of degree not greater than n. Let $U = \{t \in \mathbb{R} : |\bar{u}| \leq |\bar{y}|\}$ and $Y = \{t \in \mathbb{R} : |\bar{u}| > |\bar{y}|\}$. Sets U and Y are such that $U \cap Y = \emptyset$ and $U \cup Y = \mathbb{R}$. By Lebesgue integration theory, there exist countable closed intervals U_i, Y_i such that

$$U \subset \bigcup U_i,\ Y \subset \bigcup Y_i, \left|\left(\bigcup_i U_i\right) \cap \left(\bigcup_i Y_i\right)\right| = 0,$$

$$|U| = \sum_i |U_i|, \qquad |Y| = \sum_i |Y_i|$$

and the intervals are ordered according to $U_i < Y_i < U_{i+1} < Y_{i+1}$, where $<$ denotes the relation of left to right precedence between nonoverlapping intervals, that is $[a_1,b_1] < [a_2,b_2]$ when $b_1 \leq a_2$. By (53), $\int_{U_i} |\bar{u}(t)|\mathrm{d}t = 0$, therefore by part (a) of Lemma 2, there exist polynomials u_i of degree not exceeding n such that $B[\bar{u}](t) = u_i(t), \forall t \in U_i$. In the same way, there exist polynomials y_i such that $A[\bar{y}](t) = y_i(t), \forall t \in Y_i$. From (54), it follows that $\forall t \in U_i$, $A[\bar{y}](t) = B[\bar{u}](t) - p(t) = u_i(t) - p(t)$ and $\forall t \in Y_i$, $B[\bar{u}](t) = A[\bar{y}](t) + p(t) = y_i(t) + p(t)$. Therefore, in each interval U_i and Y_i, $A[\bar{y}](t)$ and $B[\bar{u}](t)$ are polynomials of degree less or equal than n. Consider the two consecutive intervals U_i and Y_i. Since $\int_{Y_i} |y(t)|\mathrm{d}t = 0$ and $A[\bar{y}](t)$ is a polynomial in interval U_i, then, by part (b) of Lemma 2, function $A[\bar{y}](t)$ must be equal to the same polynomial in interval Y_i, that is $\forall t \in U_i \cup Y_i$, $A[\bar{y}](t) = u_i(t) - p(t) = y_i(t)$. Analogously, as $\int_{U_{i+1}} |u(t)|\mathrm{d}t = 0$, then $\forall t \in Y_i \cup U_{i+1} B[\bar{u}](t) = u_{i+1}(t) = y_i(t) + p(t)$. By equating the two different expressions for $y_i(t)$, it follows that $u_i = u_{i+1}$ for all i. Hence, there exists one (unique) polynomial p_u such that

$$B[\bar{u}](t) = p_u(t), \quad \forall t \in \mathbb{R}. \tag{55}$$

Likewise, there exists one (unique) polynomial p_y satisfying

$$A[\bar{y}](t) = p_y(t), \quad \forall t \in \mathbb{R}. \tag{56}$$

As a consequence of (55), by Theorem 3.2.4 of Polderman and Willems (1998), it follows that, almost everywhere, $\bar{u}$ can be expressed as a linear combination of the modes $m_i^Z(t)$ associated to the zeros of (1) plus a constant term c_0, i.e., $\bar{u}(t) = c_0 + \sum_{i=1}^{m} c_i m_i^Z(t)$. Since, $\forall t \geq t_f^*$, $\bar{u}(t) = u(t) - u^*(t) = 0$ (in fact the two functions reach the same final value), $c_i = 0$, for $i = 0, \ldots, m$; then $\bar{u} = 0$ and $u(t) = u^*(t)$ almost everywhere, i.e., $\int_0^{+\infty} |u(t) - u^*(t)| \mathrm{d}t = 0$. In the same way, using relation (56) it follows that $\int_0^{+\infty} |y(t) - y^*(t)| \mathrm{d}t = 0$. □

Proposition 8. *Given a sequence of functions $(u_i, y_i) \in \mathcal{T}_{\mathbf{p}}$, if $\lim_{i \to +\infty} T_f(u_i, y_i) = t_f^*$, then $u_i \to u^*, y_i \to y^*$ in the sense of L_1 and $T_f(u^*, y^*) = t_f^*$.*

Proof. There exists a sufficiently large M such that $(u_i, y_i) \in \mathcal{T}_{\mathbf{p}}^M$ for all $i \in \mathbb{N}$. As shown in the proof of Proposition 5, $\mathcal{T}_{\mathbf{p}}^M$ is a compact set, so that it is possible to find a convergent subsequence of pairs (u_{l_i}, y_{l_i}) and its limit be denoted by $(\bar{u}, \bar{y})$. Hence $T_f(\bar{u}, \bar{y}) = t_f^*$, and by Proposition 7, it follows that $\int_0^M |\bar{u}(t) - u^*(t)| + |\bar{y}(t) - y^*(t)| \mathrm{d}t = 0$. To prove that (u_i, y_i) converges to (u^*, y^*) assume by contradiction that it does not. Then, there exists an $\epsilon > 0$ such that $\forall l > 0, \exists i_l > l : \int_0^M |u_{i_l}(t) - u^*(t)| + |y_{i_l}(t) - y^*(t)| \mathrm{d}t > \epsilon$, since $\mathcal{T}_{\mathbf{p}}^M$ is compact, it is possible to extract from the sequence with indexes $i_l, l = 1, \ldots, \infty$ a convergent subsequence, whose limit is denoted by (u_2, y_2), such that $T_f(u_2, y_2) = t_f^*$ and $\int_0^M |u_2(t) - u^*(t)| + |y_2(t) - y^*(t)| \mathrm{d}t > \epsilon$, which contradicts Proposition 7. □

Appendix B. Lemmas used in the proof of Theorem 4

Lemma 3. *Consider system Σ (1), set $T > 0$, $t_0 \in \mathbb{R}$ and consider an input–output pair $(u, y) \in \mathcal{B}$ for which $u(t) = \frac{y_f}{H(0)}, \forall t \geq t_0$ and $y(t) = \int_{-\infty}^{t} h(t - \tau) u(\tau) \mathrm{d}\tau$ satisfies $y(t_0 + kT) = y_f$, for $k = 0, \ldots, n-1$. Moreover, assume that the distinct roots $p_1, \ldots, p_l$ of the polynomial $s^n + a_{n-1}s^{n-1} + \cdots + a_0$ satisfy $p_i - p_r \neq k\frac{2\pi \mathrm{j}}{T}, \forall i, r = 1, \ldots, l, \forall k \in \mathbb{Z} - \{0\}$ where j denotes the imaginary unit. Then it follows that $y(t) = y_f, \forall t \geq t_0$.*

The proof is based on the properties of the generalized Vandermonde matrix. This proof and those of the next two technical Lemmas have been omitted for sake of brevity.

Lemma 4. *Consider system Σ (1) and an input–output pair $(u, y) \in \mathcal{B}$ for which u is constant in the intervals $[kT, (k+1)T[, \forall k \in \mathbb{Z}$ and $y(t) = \int_{-\infty}^{t} h(t - \tau) u(\tau) \mathrm{d}\tau$ satisfies $y(kT) \in [y_c^-, y_c^+], \forall k \in \mathbb{Z}$. Then $\forall t \in \mathbb{R}$*

$$\begin{aligned} y(t) - y_c^+ &\leq T \left(|h(0^+)| + \|Dh(\cdot)\|_1 \right) \|u(\cdot)\|_\infty, \\ y_c^- - y(t) &\leq T \left(|h(0^+)| + \|Dh(\cdot)\|_1 \right) \|u(\cdot)\|_\infty. \end{aligned} \tag{57}$$

Lemma 5. *Consider system Σ (1). Given $\epsilon > 0$, there exist two positive constants M_u, M_y, such that for any vector $\mathbf{z} = [z_0, z_1, \ldots, z_{n-1}]^T \in \mathbb{R}^n$, and any $u_f, a \in \mathbb{R}$, there exists an input–output pair $(\tilde{u}(t), \tilde{y}(t)) \in \mathcal{B} \cap C^n$ such that*

(1) $\tilde{u}(t) = 0, \forall t \leq a$ *and* $\tilde{u}(t) = u_f, \forall t \geq a + \epsilon$;
(2) $\tilde{y}(t) = 0, \forall t \leq a, \tilde{y}(a + \epsilon) = z_0, D\tilde{y}(a + \epsilon) = z_1, \ldots, D^{n-1}\tilde{y}(a + \epsilon) = z_{n-1}$;
(3) $\|\tilde{u}\|_\infty \leq M_u(\|\mathbf{z}\| + |u_f|), \|\tilde{y}\|_\infty \leq M_y(\|\mathbf{z}\| + |u_f|)$.

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