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Worst-Case Optimal Static Output Feedback for Uncertain Systems

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Abstract. This paper proposes a worst-case optimal approach to the synthesis of a static output feedback for a linear, time-invariant, multivariable system depending on uncertain parameters which nonlinearly affect a given state-space model. The aim is to seek an output-to-input feedback matrix that robustly stabilizes the closed-loop system while minimizing, over the uncertain parameter domain, the (worst-case) maximum of a composite quadratic index. The emerging minimax problem is shown to be exactly equivalent to a semi-infinite optimization problem for which an estimate of a global solution is obtained through a genetic/interval algorithm. This is a hybrid algorithm that combines a genetic algorithm (at the upper level) with an interval procedure (acting at the lower level). Computational results for a two-input two-output (TITO) system are included.

Keywords: static output feedback, worst-case design, semi-infinite optimization, hybrid algorithms

1. Introduction

In the context of linear, time-invariant systems the problem of static output feedback is a long-standing one of control theory. In spite of the large amount of research contributions, cf. the survey of Syrmos et al. (1997), it is somewhat considered an open problem which deserves further investigation. Indeed, all the necessary and sufficient conditions emerged in the literature do not appear efficiently testable and practical numerical methods till now do not offer global convergence to the solution set. The problem is more complex in the case of uncertain systems and, as a consequence, a few results are available. For example, Cao et al. (1998) apply an iterative linear matrix inequality approach to guarantee the robust (H_∞) stability of a plant through a static output feedback. Analogously, the paper of Varga and Pieters (1998) deals with the robust stability of discrete-time periodic systems, i.e. systems whose matrices of the state-space model change periodically. The target is to stabilize the uncertain system while minimizing an assigned performance index. The algorithm they propose has local convergence properties being based on a gradient-search approach and requires the preliminary selection of a feasible starting point. The solution is a periodic output-feedback matrix. The problem is even hardened when a static output-feedback matrix has to be used to stabilize an uncertain plant. This is the case considered in

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the paper of Sreeram et al. (1996): a (finite) set of continuous-time plants is simultaneously stabilized by means of a single fixed-structure controller. The problem is solved using an LQ approach aiming to simultaneously achieve stability while minimizing a performance index. The solution proposed seems to be very efficient because the algorithm, starting from a feasible point, finds the minimizer with a gradient flow approach that does not need to check the system stability during the iterations. Also in this case a local solution is found. Similarly, in Liu et al. (1996) an optimal simultaneous stabilization problem is solved for a finite set of linear continuous time-invariant descriptor systems.

A more general problem concerns systems with parametric uncertainties: an infinite number of plants are simultaneously stabilized by considering a static output feedback. An optimal solution with a quantifier elimination technique was presented by Abdallah et al. (1996). Still considering parametric uncertainties, İftar and Özgüner (1996) proposed a game theoretic approach to the optimal synthesis of a static output feedback. Taking into account uncertainties affinely entering the system matrix of a state-space model, they sought a stabilizing feedback for the closed-loop over the uncertain parameter set while minimizing a quadratic performance index (associated to a given initial state) under the worst perturbations.

In this paper we generalize the problem of İftar and Özgüner by considering parametric uncertainties that nonlinearly affect the system matrix. The problem is solved using a global optimization approach. Specifically, for a multi-input multi-output (MIMO) system a static output feedback that robustly stabilizes the closed-loop is found. At the same time, the (worst-case) maximum, over all possible uncertain parameters, of a composite performance index is minimized. This is the trace of a positive semi-definite matrix associated to a given quadratic integral. Benefiting from appropriate assumptions and from a suitable reformulation of the Liénard-Chipard criterion, the posed synthesis problem is shown to be equivalent to a semi-infinite optimization problem for which an estimate of the global solution is obtained through a recently devised genetic/interval algorithm (Guarino Lo Bianco and Piazzi, 2001a, 2001b). This hybrid algorithm combines a genetic approach, based on the evolutionary paradigm, with an interval procedure, i.e. a branch-and-bound algorithm based on concepts of Interval Analysis (Jaulin et al., 2001; Moore, 1979). The rationale lies in using the interval procedure to evaluate the semi-infinite constraints and to convert the semi-infinite optimization problem into a finite one to be solved by the genetic algorithm. The purpose of the interval procedure is to verify deterministically the feasibility of the solution. Indeed, in this paper it will be shown how the stability requirement can be converted into semi-infinite constraints for the optimization problem so that, due to the interval approach, it is possible to claim the certain robust stability of the closed-loop plant. The use of a genetic algorithm is justified by the problem at hand which is nonlinear and nonconvex: the stochastic approach is adopted to obtain a globally convergent algorithm with a moderate time burden. Indeed, a drawback of the genetic/interval algorithm is the evaluation time. Owing to the branch-and-bound strategy of the interval procedure only a moderate number of uncertain parameters (up to 4 or 5) can be processed. This normally suffices to cover many significant control problems. As the number of uncertain parameters increases other strategies can be used to guarantee the robust stability. For example, the same genetic/interval algorithm has been adopoted in Lo Bianco and Piazzi (1998, 2000) for the H_2/H_∞ control of scalar plants.

Other algorithms for global semi-infinite optimization can be fruitfully used to solve the proposed control problem by means of the modellization presented in this paper. However, many available algorithms (see e.g. (Lawrence and Tits, 1998; Zhou and Tits, 1996)) verify the feasibility of the solution by discretizing the semi-infinite constraints. These algorithms cannot guarantee with certainty the closed-loop robustness because, owing the constraints discretization, the stability is verified over a finite set of sampled plants.

The paper is organized as follows. Section 2 formally states the problem and shows (Proposition 2) its equivalence to a semi-infinite minimax optimization problem. In Section 3, this is transformed into a (standard) semi-infinite problem and then reduced, using a penalty method, into an unconstrained finite minimization problem. A synthesis example for a two-input two-output (TITO) system is exposed and solved in Section 4. Final remarks are reported in Section 5.

Let us introduce some notation used in the paper. The real n-dimensional space is denoted by $\mathbb{R}^n$, the space of $n \times m$ matrixes by $\mathbb{R}^{n\times m}$, the set of strictly positive real numbers by $\mathbb{R}^+$, and the open left-hand complex plane by $\mathbb{C}^-$. The spectrum of eigenvalues of $A \in \mathbb{R}^{n\times n}$ is denoted by $\sigma(A)$ and the transpose of a matrix A is indicated by A^T. The column stacking operator is vec: $\mathbb{R}^{n\times m} \to \mathbb{R}^{nm}$ and the trace operator is tr: $\mathbb{R}^{n\times n} \to \mathbb{R}$. The set union with repetition is denoted by $\dot{\cup}$.

2. Worst-case optimal output feedback

Consider the MIMO system $\Sigma(\mathbf{p})$ described with the following state-space model:

$$\Sigma(\mathbf{p}) := \begin{cases} \dot{x}(t) = A(\mathbf{p})x(t) + Bu(t); \quad x(0) = x_0 \\ y(t) = Cx(t) \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^l$ is the control input, $y \in \mathbb{R}^q$ is the system output, and $\mathbf{p} \in \mathbb{R}^m$ is the uncertain parameter vector belonging to the given multidimensional interval $\mathcal{P} := [p_1^-, p_1^+] \times [p_2^-, p_2^+] \times \cdots \times [p_m^-, p_m^+]$. The matrixes appearing in (1) are of compatible dimensions and all entries of $A(\mathbf{p})$ are supposed to be continuously differentiable nonlinear real functions over the domain $\mathcal{P}$.

Roughly speaking, we seek a static output feedback $u = Ky$, $K \in \mathbb{R}^{l\times q}$ that optimally stabilizes, according to a worst-case quadratic index, the whole uncertain system $\Sigma(\mathbf{p})$. Specifically consider the quadratic integral

$$J := \int_0^\infty (x^T Qx + u^T Ru)\, dt \tag{2}$$

where Q and R are given symmetric matrixes respectively positive-semidefinite and positive-definite. Denote by d the rank of Q and introduce the full rank matrix $D \in \mathbb{R}^{d\times n}$ such that $Q = D^T D$. Also introduce the non singular matrix $L \in \mathbb{R}^{l\times l}$ such that $R = L^T L$. Define the vector variables $\xi = [\xi_1\ \xi_2 \ldots \xi_d]^T := Dx$ and $\eta = [\eta_1\ \eta_2 \ldots \eta_l]^T := Lu$. Hence integral (2) is equivalently given by

$$J = \sum_{i=1}^{d} \int_0^\infty \xi_i(t)^2\, dt + \sum_{i=1}^{l} \int_0^\infty \eta_i(t)^2\, dt. \tag{3}$$

Integral (2) is associated to the state evolution of the closed-loop system:

$$\begin{cases} \dot{x} = (A(\mathbf{p}) + BKC)x; \quad x(0) = x_0 \\ u = KCx. \end{cases} \tag{4}$$

Provided that K is a stabilizing feedback, the integral (2) or (3) is well-defined and finite and depends on the vector parameter $\mathbf{k} := \mathrm{vec}(K) \in \mathbb{R}^{lq}$, on $\mathbf{p} \in \mathcal{P}$, and on the initial state $x_0 \in \mathbb{R}^n$. These functional dependencies can be evidenced by denoting the integral as $J(\mathbf{k}, \mathbf{p}, x_0)$. Moreover, $J(\mathbf{k}, \mathbf{p}, x_0)$ is a quadratic form of x_0 (Anderson and Moore, 1990, p. 317)

$$J(\mathbf{k}, \mathbf{p}, x_0) = x_0^T P x_0 \tag{5}$$

where $P = P^T$ is the unique solution of the Lyapunov equation

$$P(A(\mathbf{p}) + BKC) + (A(\mathbf{p})^T + C^T K^T B^T)P + C^T K^T RKC + Q = 0. \tag{6}$$

Matrix $P = P(\mathbf{k}, \mathbf{p})$ is positive-semidefinite for all $\mathbf{p} \in \mathcal{P}$ provided that $A(\mathbf{p}) + BKC$ is stable for all $\mathbf{p} \in \mathcal{P}$. Taking into account a technical standpoint, we require that the absolute values of all entries of $\mathbf{k}$ be bounded by a sufficiently large $\bar{k}$. The following assumption is then necessary.

Assumption 1. Let define $\mathcal{K} := [-\bar{k}, \bar{k}] \times [-\bar{k}, \bar{k}] \times \cdots \times [-\bar{k}, \bar{k}] \subseteq \mathbb{R}^{lq}$ and formally denote by $K(\mathbf{k})$ the feedback matrix. There exists a design parameter vector $\mathbf{k} \in \mathcal{K}$ such that

$$\sigma(A(\mathbf{p}) + BK(\mathbf{k})C) \subseteq \mathbb{C}^- \quad \forall \mathbf{p} \in \mathcal{P}. \tag{7}$$

Remark 1. The validity of Assumption 1 can be checked, without any ad hoc hypothesis, by means of the solution method proposed in the next section.

If the initial state x_0 is known, the output feedback problem should be centered on minimizing the worst-case index (5) subject to the robust closed-loop stability. However in the usual practical cases x_0 is not known, so that a possible choice, in searching an optimal output feedback, is to find a robustly stabilizing $\mathbf{k}^* \in \mathcal{K}$ that globally minimizes the worst-case index given by the trace of $P(\mathbf{k}, \mathbf{p})$ (Anderson and Moore, 1990). Formally we have to solve:

$$\inf_{\mathbf{k} \in \mathcal{K}} \max_{\mathbf{p} \in \mathcal{P}} \{\mathrm{tr}(P(\mathbf{k}, \mathbf{p}))\} \tag{8}$$

subject to

$$\sigma(A(\mathbf{p}) + BK(\mathbf{k})C) \subseteq \mathbb{C}^- \quad \forall \mathbf{p} \in \mathcal{P}. \tag{9}$$

Let us introduce the characteristic polynomial of matrix $A(\mathbf{p}) + BK(\mathbf{k})C$:

$$\begin{aligned}\chi(s; \mathbf{k}, \mathbf{p}) &:= \det(sI - A(\mathbf{p}) - BK(\mathbf{k})C) \\ &= s^n + \chi_{n-1}(\mathbf{k}, \mathbf{p})s^{n-1} + \cdots + \chi_1(\mathbf{k}, \mathbf{p})s + \chi_0(\mathbf{k}, \mathbf{p}).\end{aligned} \tag{10}$$

Denote by $H_i(\mathbf{k}, \mathbf{p})$ the i-th order Hurwitz determinant associated to $\chi(s; \mathbf{k}, \mathbf{p})$. From Liénard and Chipard's criterion (Gantmacher, 1959) it can be easily deduced the following result.

Property 1. *The matrix $A(\mathbf{p}) + BK(\mathbf{k})C$ is stable for all $\mathbf{p} \in \mathcal{P}$ if and only if the following n semi-infinite inequalities hold*

$$\begin{aligned}\chi_0(\mathbf{k}, \mathbf{p}) > 0 \quad \forall \mathbf{p} \in \mathcal{P} \\ \chi_1(\mathbf{k}, \mathbf{p}) > 0, \chi_3(\mathbf{k}, \mathbf{p}) > 0, \ldots, \chi_v(\mathbf{k}, \mathbf{p}) > 0 \quad \forall \mathbf{p} \in \mathcal{P} \\ H_{n-1}(\mathbf{k}, \mathbf{p}) > 0, H_{n-3}(\mathbf{k}, \mathbf{p}) > 0, \ldots, H_w(\mathbf{k}, \mathbf{p}) > 0 \quad \forall \mathbf{p} \in \mathcal{P}\end{aligned}$$

with $v = n - 1$, $w = 3$ if n is even, while $v = n - 2$, $w = 2$ if n is odd.

A stronger stability result that involve just two semi-infinite inequalities is given in the following proposition.

Proposition 1. *The matrix $A(\mathbf{p}) + BK(\mathbf{k})C$ is stable for all $\mathbf{p} \in \mathcal{P}$ if and only if:*

$$\chi_0(\mathbf{k}, \mathbf{p}) > 0, H_{n-1}(\mathbf{k}, \mathbf{p}) > 0, \quad \forall \mathbf{p} \in \mathcal{P} \tag{11}$$

$$\chi_1(\mathbf{k}, \mathbf{p}_0) > 0, \chi_3(\mathbf{k}, \mathbf{p}_0) > 0, \ldots, \chi_v(\mathbf{k}, \mathbf{p}_0) > 0 \tag{12}$$

$$H_{n-3}(\mathbf{k}, \mathbf{p}_0) > 0, H_{n-5}(\mathbf{k}, \mathbf{p}_0) > 0, \ldots, H_w(\mathbf{k}, \mathbf{p}_0) > 0 \tag{13}$$

with $v = n - 1$, $w = 3$ if n is even, while $v = n - 2$, $w = 2$ if n is odd and $\mathbf{p}_0$ is any chosen point in $\mathcal{P}$.

Proof: The necessity part of the proof is a straightforward consequence of Property 1. To demonstrate the sufficiency let us assume by a contradiction argument that there exists a $\mathbf{p}' \in \mathcal{P}$ such that the matrix $A(\mathbf{p}') + BK(\mathbf{k})C$ is not stable. Then define a curve γ, parameterized by $\lambda \in [0, 1]$, joining the two points $\mathbf{p}'$ and $\mathbf{p}_0$, and whose path is completely contained in $\mathcal{P}$:

$$\gamma(\lambda) : [0, 1] \to \mathbb{R}^m \text{ with } \gamma(0) = \mathbf{p}_0,\ \gamma(1) = \mathbf{p}' \text{ and } \gamma([0, 1]) \subseteq \mathcal{P}.$$

The coefficients of the polynomial $\chi(s; \mathbf{k}, \mathbf{p})$ are continuous functions of $\mathbf{p}$, thus its roots denoted as $z_i(\mathbf{k}, \mathbf{p})$, $i = 1, \ldots, n$ are continuous on $\mathbf{p}$. The real part of at least one root of the characteristic polynomial $\chi(s; \mathbf{k}, \mathbf{p}')$ is not negative while, according with (11)–(13), polynomial $\chi(s; \mathbf{k}, \mathbf{p}_0)$ is Hurwitz so that the real parts of all its roots are strictly negative. By virtue of the continuity property of $\chi(s; \mathbf{k}, \mathbf{p})$, there exists a $\lambda_c \in [0, 1]$ such that $\chi(s; \mathbf{k}, \gamma(\lambda_c))$ has at least one root with the real part equal to zero and $\gamma(\lambda_c) \in \mathcal{P}$. Two possible situations could arise: this root is real and equal to zero and in this case from (10) we evidently have $\chi_0(\mathbf{k}, \gamma(\lambda_c)) = 0$ or the characteristic polynomial has two imaginary roots. In the latter case $H_{n-1}(\mathbf{k}, \gamma(\lambda_c)) = 0$ because of Orlando's formula (Gantmacher, 1959) which expresses H_{n-1} as a function of the polynomial roots:

$$H_{n-1}(\mathbf{k}, \gamma(\lambda_c)) = (-1)^{\frac{n(n-1)}{2}} \prod_{i<k}^{1,\ldots,n} (z_i(\mathbf{k}, \gamma(\lambda_c)) + z_k(\mathbf{k}, \gamma(\lambda_c))).$$

In conclusion, for $\mathbf{p} = \gamma(\lambda_c)$ at least one of the two inequalities (11) is violated. □

A drawback of problem formulation (8) is the possible presence of a degenerate solution, i.e. $\mathbf{k}^*$ may belong to the boundary of the feasible set

$$\{\mathbf{k} \in \mathcal{K} : \sigma(A(\mathbf{p}) + BK(\mathbf{k})C) \subseteq \mathbb{C}^- \ \forall \mathbf{p} \in \mathcal{P}\}.$$

If this is the case, there exist $\mathbf{p}^* \in \mathcal{P}$ such that matrix $A(\mathbf{p}^*) + BK(\mathbf{k}^*)C$ is not (Hurwitz) stable. More precisely (at least) one eigenvalue of $A(\mathbf{p}^*) + BK(\mathbf{k}^*)C$ lies on the imaginary axis. In order to avoid this possible degeneracy we include this further assumption.

Assumption 2. There exists a matrix $M \in \mathbb{R}^{n\times d}$ such that

$$\sigma(A(\mathbf{p}) + MD) \subseteq \mathbb{C}^- \ \forall \mathbf{p} \in \mathcal{P}. \tag{14}$$

Property 2. *On Assumption* 2, *the pair* $(D, A(\mathbf{p}))$ *is detectable* $\forall \mathbf{p} \in \mathcal{P}$.

Remark 2. It is worth noting that the detectability of $(D, A(\mathbf{p}))$ $\forall \mathbf{p} \in \mathcal{P}$ does not imply the statement given in (14). On the other hand, Assumption 2 in general is simpler to check that detectability of $(D, A(\mathbf{p}))$ $\forall \mathbf{p} \in \mathcal{P}$. Indeed, analogously to Assumption 1, it can be verified by means of a trivial modification of the solution method proposed in the next section.

An useful, yet straightforward, result to be used in the sequel is the following.

Lemma 1. *Let be given a matrix* $A \in \mathbb{R}^{n\times n}$ *and the associated system* $\dot{x}(t) = Ax(t)$, $x(0) = x_{0i}$ *with* $\{x_{01}, x_{02}, \ldots, x_{0n}\}$ *being any vector base of the state-space* $\mathbb{R}^n$. *The spectrum of eigenvalues* $\sigma(A)$ *is stable if and only if*

$$\lim_{t\to\infty} x(t) = 0 \quad \forall i \in \{1, \ldots, n\}. \tag{15}$$

In the following we choose $\mathbf{p}_0 := \mathrm{mid}(\mathcal{P})$, i.e. $\mathbf{p}_0$ is the midpoint of the "box" $\mathcal{P}$.

Proposition 2. *On Assumptions* 1 *and* 2, *there exists a sufficiently small* $\varepsilon \in \mathbb{R}^+$ *for which optimization problem* (8) *is equivalent to*

$$\min_{\mathbf{k}\in\mathcal{K}} \max_{\mathbf{p}\in\mathcal{P}} \{\mathrm{tr}(P(\mathbf{k}, \mathbf{p}))\} \tag{16}$$

subject to

$$\chi_0(\mathbf{k},\mathbf{p}) \geq \varepsilon, \quad H_{n-1}(\mathbf{k},\mathbf{p}) \geq \varepsilon \quad \forall \mathbf{p} \in \mathcal{P}$$
$$\chi_1(\mathbf{k},\mathbf{p}_0) \geq \varepsilon, \quad \chi_3(\mathbf{k},\mathbf{p}_0) \geq \varepsilon, \ldots, \chi_v(\mathbf{k},\mathbf{p}_0) \geq \varepsilon$$
$$H_{n-3}(\mathbf{k},\mathbf{p}_0) \geq \varepsilon, \quad H_{n-5}(\mathbf{k},\mathbf{p}_0) \geq \varepsilon, \ldots, H_w(\mathbf{k},\mathbf{p}) \geq \varepsilon$$

Proof: Assumption 1 assures that problem (8) always admits a solution; denote it by $\mathbf{k}^*$. First we show that $K(\mathbf{k}^*)$ is a robustly stabilizing feedback, i.e. $\sigma(A(\mathbf{p}) + BK(\mathbf{k}^*)C) \subseteq \mathbb{C}^- \ \forall \mathbf{p} \in \mathcal{P}$. There exists $\mathbf{p}^* \in \mathcal{P}$ such that the optimal performance is

$$\begin{aligned} \operatorname{tr}(P(\mathbf{k}^*,\mathbf{p}^*)) &= J(\mathbf{k}^*,\mathbf{p}^*,e_1) + \cdots + J(\mathbf{k}^*,\mathbf{p}^*,e_n) \\ &= \max_{\mathbf{p}\in\mathcal{P}}\{J(\mathbf{k}^*,\mathbf{p},e_1) + \cdots + J(\mathbf{k}^*,\mathbf{p},e_n)\} \end{aligned} \tag{17}$$

where e_i, $i \in \hat{n} := \{1, 2, \ldots, n\}$ are the unit-vectors of $\mathbb{R}^n$. By considering the finiteness of $\operatorname{tr}(P(\mathbf{k}^*,\mathbf{p}^*))$ along with expression (17) we imply that

$$J(\mathbf{k}^*,\mathbf{p},e_i) \text{ is finite } \forall \mathbf{p} \in \mathcal{P}\ \forall i \in \hat{n}. \tag{18}$$

Hence, by virtue of (3), we infer

$$\lim_{t\to\infty} \xi(t) = 0\ \forall \mathbf{p} \in \mathcal{P}\ \forall i \in \hat{n}, \tag{19}$$
$$\lim_{t\to\infty} \eta(t) = 0\ \forall \mathbf{p} \in \mathcal{P}\ \forall i \in \hat{n}. \tag{20}$$

Property 2 assures that pair $(D, A(\mathbf{p}))$ is detectable $\forall \mathbf{p} \in \mathcal{P}$. Therefore

$$\left(\begin{bmatrix} D \\ K(\mathbf{k}^*)C \end{bmatrix}, A(\mathbf{p})\right) \text{ is detectable } \forall \mathbf{p} \in \mathcal{P}.$$

For any matrix $M \in \mathbb{R}^{n\times(d+l)}$ also pair

$$\left(\begin{bmatrix} D \\ K(\mathbf{k}^*)C \end{bmatrix}, A(\mathbf{p}) + M\begin{bmatrix} D \\ K(\mathbf{k}^*)C \end{bmatrix}\right)$$

is detectable $\forall \mathbf{p} \in \mathcal{P}$. In particular choose $M := [0\ B]$ and obtain that

$$\left(\begin{bmatrix} D \\ K(\mathbf{k}^*)C \end{bmatrix}, A(\mathbf{p}) + BK(\mathbf{k}^*)C\right) \text{ is detectable } \forall \mathbf{p} \in \mathcal{P}.$$

In order to simplify the notation let us define $\hat{A} := A(\mathbf{p}) + BK(\mathbf{k}^*)C$, $\hat{D} := [D^T\ C^T K(\mathbf{k}^*)^T]^T$, and $\hat{y} := [\xi^T\ u^T]^T$. By virtue of (19) and (20) we have

$$\lim_{t\to\infty} \hat{y}(t) = 0\ \forall \mathbf{p} \in \mathcal{P}\ \forall i \in \hat{n}, \tag{21}$$

where $\hat{y}(t)$ is the output of the following detectable system

$$\begin{cases} \dot{x}(t) = \hat{A}x(t); \quad x(0) = e_i \\ \hat{y}(t) = \hat{D}x(t). \end{cases} \tag{22}$$

Perform a state-space coordinate transformation $x = Tz$ where $T := [T_1\ T_2]$ is a nonsingular matrix, with $\mathrm{im}(T_1)$ being the unobservable subspace of (22). In the new coordinate state-variables, system (22) becomes

$$\begin{cases} \dot{z}(t) = A'z(t); \quad z(0) = T^{-1}e_i \\ \hat{y}(t) = D'z(t) \end{cases} \tag{23}$$

where matrixes

$$A' = \begin{bmatrix} A'_1 & A'_3 \\ 0 & A'_2 \end{bmatrix}, \quad D' = [0\ \ D'_2] \tag{24}$$

are partitioned according to the structure of T. Evidently, it holds

$$\sigma(A') = \sigma(A'_1)\ \dot{\cup}\ \sigma(A'_2) \tag{25}$$

with $\sigma(A'_1) \in \mathbb{C}^-$ and pair (D'_2, A'_2) being (completely) observable. According to the partitioning of A' and T designate $z := [z_1^T\ \ z_2^T]^T$ with $z_2 \in \mathbb{R}^{n_2}$ and $T^{-1} := [S_1^T\ \ S_2^T]^T$. Thus $\hat{y}(t)$ is also the output of the system

$$\begin{cases} \dot{z}_2(t) = A'_2 z_2(t); \quad z_2(0) = S_2 e_i \\ \hat{y}(t) = D'_2 z_2(t). \end{cases} \tag{26}$$

Denoting by $\hat{y}^{(j)}(t)$ the j-th derivative of $\hat{y}(t)$, from (21), obtain

$$\lim_{t\to\infty} \hat{y}^{(j)}(t) = 0 \ \ \forall j \in \mathbb{N}, \forall \mathbf{p} \in \mathcal{P}, \ \text{and}\ \forall i \in \hat{n}. \tag{27}$$

By computing the derivative $\hat{y}^{(j)}(t)$ from model (26) we have

$$\begin{bmatrix} \hat{y}(t) \\ \hat{y}^{(1)}(t) \\ \vdots \\ \hat{y}^{(n_2-1)}(t) \end{bmatrix} = \begin{bmatrix} D'_2 \\ D'_2 A'_2 \\ \vdots \\ D'_2 A_2'^{(n_2-1)} \end{bmatrix} z_2(t),$$

where in the right hand it appears the full rank observability matrix which we denote by Γ. Thus $z_2(t) = \Gamma^{\#}[\hat{y}(t)\ \hat{y}^{(1)}(t)\ldots\hat{y}^{(n_2-1)}(t)]^T$ where $\Gamma^{\#}$ denotes the Moore-Penrose pseudoinverse of Γ. From (27) deduce

$$\lim_{t\to\infty} z_2(t) = 0 \ \ \forall \mathbf{p} \in \mathcal{P}, \ \text{and}\ \forall i \in \hat{n}. \tag{28}$$

Taking into account that S_2 is a full rank matrix and evidently $n_2 \leq n$, it follows that there exists among vectors $S_2 e_i$, $i \in \hat{n}$ a subset of n_2 linear independent vectors of $\mathbb{R}^{n_2}$. Hence, by virtue of Lemma 1, we have proved that $\sigma(A_2') \subseteq \mathbb{C}^- \ \forall \mathbf{p} \in \mathcal{P}$ so that by (25) it holds

$$\sigma(A(\mathbf{p}) + BK(\mathbf{k}^*)C) \subseteq \mathbb{C}^- \quad \forall \mathbf{p} \in \mathcal{P}. \tag{29}$$

This robust stability result implies that problem (8) can be equivalently replaced by

$$\min_{\mathbf{k} \in \mathcal{K}} \max_{\mathbf{p} \in \mathcal{P}} \{\mathrm{tr}(P(\mathbf{k}, \mathbf{p}))\} \tag{30}$$

subject to the inequalities of Proposition 2. Now choose any positive ε smaller or equal to these values:

$$\begin{gathered} \min_{\mathbf{p} \in \mathcal{P}} \chi_0(\mathbf{k}^*, \mathbf{p}), \quad \min_{\mathbf{p} \in \mathcal{P}} H_{n-1}(\mathbf{k}^*, \mathbf{p}), \\ \chi_1(\mathbf{k}^*, \mathbf{p}_0), \quad \chi_3(\mathbf{k}^*, \mathbf{p}_0), \ldots, \chi_v(\mathbf{k}^*, \mathbf{p}_0), \\ H_{n-3}(\mathbf{k}^*, \mathbf{p}_0), \quad H_{n-5}(\mathbf{k}^*, \mathbf{p}_0), \ldots, H_w(\mathbf{k}^*, \mathbf{p}_0). \end{gathered}$$

This choice of ε determines that the set of global minimizers and associated global minimum of problem (16) is exactly the same of problem (30). □

Remark 3. In the next section, problem (16) will be the starting point of our numerical approach to solving (8). From the practical side, a proper choice of ε in not an issue: just choose the smallest real number compatible with the implemented computational precision.

3. The semi-infinite optimization approach

The minimax problem (16) can be conveniently converted into a standard semi-infinite optimization problem by augmenting the design parameter vector: $\mathbf{k}_a := [\mathbf{k}^T k_{lq+1}]^T \in \mathcal{K} \times \mathbb{R}^+$. Indeed, problem (16) is equivalent to

$$\min_{\mathbf{k}_a \in \mathcal{K} \times \mathbb{R}^+} k_{lq+1} \tag{31}$$

subject to

$$\begin{gathered} \chi_0(\mathbf{k}, \mathbf{p}) \geq \varepsilon, \ H_{n-1}(\mathbf{k}, \mathbf{p}) \geq \varepsilon, \ \forall \mathbf{p} \in \mathcal{P} \\ \chi_1(\mathbf{k}, \mathbf{p}_0) \geq \varepsilon, \ \chi_3(\mathbf{k}, \mathbf{p}_0) \geq \varepsilon, \ldots, \chi_v(\mathbf{k}, \mathbf{p}_0) \geq \varepsilon \\ H_{n-3}(\mathbf{k}, \mathbf{p}_0) \geq \varepsilon, \ H_{n-5}(\mathbf{k}, \mathbf{p}_0) \geq \varepsilon, \ldots, H_w(\mathbf{k}, \mathbf{p}_0) \geq \varepsilon \\ \mathrm{tr}(P(\mathbf{k}, \mathbf{p})) \leq k_{lq+1} \quad \forall \mathbf{p} \in \mathcal{P}. \end{gathered}$$

To compute the $\mathrm{tr}(P(\mathbf{k}, \mathbf{p}))$ it should be possible to solve the Lyapunov Eq. (6) which is actually a linear algebraic system of equations whose order is $n(n+1)/2$. However, for the problem at hand, it appears simpler its reduction to a collection of scalar H_2 norm computations (of order n) performed through the method proposed by Katz (1952). Consider the following single-input multi-output (SIMO) system

$$\begin{cases} \dot{x}(t) = (A(\mathbf{p}) + BK(\mathbf{k})C)x(t) + x_0\psi(t); \quad x(0) = 0 \\ \xi(t) = Dx(t) \\ \eta(t) = LK(\mathbf{k})Cx(t). \end{cases} \tag{32}$$

If we apply an unit impulse $\delta(t)$ to the input of system (32) then its state evolution $x(t)$ is the same of system (4). Thus, all terms of (3) can be determined as follows

$$\int_0^\infty \xi_i(t)^2\, dt = \left\| T_{\xi_i\psi}(s; \mathbf{k}, \mathbf{p}, x_0) \right\|_2^2, \quad i = 1, \ldots, d \tag{33}$$

$$\int_0^\infty \eta_i(t)^2\, dt = \left\| T_{\eta_i\psi}(s; \mathbf{k}, \mathbf{p}, x_0) \right\|_2^2, \quad i = 1, \ldots, l \tag{34}$$

where $T_{\xi_i\psi}(s; \mathbf{k}, \mathbf{p}, x_0)$ and $T_{\eta_i\psi}(s; \mathbf{k}, \mathbf{p}, x_0)$ are respectively the transfer functions from ψ to ξ_i and from ψ to η_i. Then, the following formula can be deduced

$$\operatorname{tr}(P(\mathbf{k}, \mathbf{p})) = \sum_{i=1}^{n} \sum_{j=1}^{d} \left\| T_{\xi_j\psi}(s; \mathbf{k}, \mathbf{p}, e_i) \right\|_2^2 + \sum_{i=1}^{n} \sum_{j=1}^{l} \left\| T_{\eta_j\psi}(s; \mathbf{k}, \mathbf{p}, e_i) \right\|_2^2, \tag{35}$$

with e_i, $i = 1, \ldots, n$ being the unit-vectors of $\mathbb{R}^n$.

The search domain $\mathcal{K} \times \mathbb{R}^+$ of problem (31) can be restricted into a bounded multidimensional interval of $\mathbb{R}^{lq+1}$ if an upper bound k_{lq+1}^+ of k_{lq+1}^* is known, with $\mathbf{k}_a^* := [(\mathbf{k}^*)^T\ k_{lq+1}^*]^T$ being a global minimizer of (31). Indeed, $\mathcal{K}_a := \mathcal{K} \times [0, k_{lq+1}^+]$ can replace $\mathcal{K} \times \mathbb{R}^+$ in problem (31). At this moment, let us suppose that k_{lq+1}^+ is available. Optimization problem (31) can be converted into an unconstrained finite problem by using the following penalty method:

$$\min_{\mathbf{k}_a \in \mathcal{K}_a} \left\{ k_{lq+1} + \sum_{i=1}^{n+1} \Phi(\lambda_i(\mathbf{k}_a)) \right\} \tag{36}$$

where

$$\lambda_1(\mathbf{k}_a) := \min_{\mathbf{p} \in \mathcal{P}} \{\chi_0(\mathbf{k}, \mathbf{p}) - \varepsilon\}$$

$$\lambda_2(\mathbf{k}_a) := \min_{\mathbf{p} \in \mathcal{P}} \{H_{n-1}(\mathbf{k}, \mathbf{p}) - \varepsilon\}$$

$$\lambda_i(\mathbf{k}_a) := \begin{cases} \chi_{2i-5}(\mathbf{k}, \mathbf{p}_0) - \varepsilon & \text{if } i = 3, 4, \ldots, j \\ H_{2i-n-3}(\mathbf{k}, \mathbf{p}_0) - \varepsilon & \text{if } i = j+1, j+2, \ldots, n \end{cases}$$

with $j = (n+4)/2$ if n is even and $j = (n+3)/2$ if n is odd;

$$\lambda_{n+1}(\mathbf{k}_a) := \min_{\mathbf{p} \in \mathcal{P}} \{k_{lq+1} - \operatorname{tr}(P(\mathbf{k}, \mathbf{p}))\}.$$

The penalty function $\Phi(\lambda)$ is given by

$$\Phi(\lambda) := \begin{cases} 0 & \text{if } \lambda \in [0, +\infty) \\ -(M/T)\lambda & \text{if } \lambda \in [-T, 0) \\ M & \text{if } \lambda \in (-\infty, -T) \end{cases} \tag{37}$$

The solution accuracy directly depends on the choice of both M and T. To obtain more precise solutions it is necessary to increase M and (or) to decrease T.

This two-step procedure can be used to calculate k^+_{lq+1}.

1. Find any $\mathbf{k}' \in \mathcal{K}$ such that

$$\chi_0(\mathbf{k}', \mathbf{p}) \geq \varepsilon,\ H_{n-1}(\mathbf{k}', \mathbf{p}) \geq \varepsilon,\ \forall \mathbf{p} \in \mathcal{P}$$
$$\chi_1(\mathbf{k}', \mathbf{p}_0) \geq \varepsilon,\ \chi_3(\mathbf{k}', \mathbf{p}_0) \geq \varepsilon, \ldots, \chi_v(\mathbf{k}', \mathbf{p}_0) \geq \varepsilon$$
$$H_{n-3}(\mathbf{k}', \mathbf{p}_0) \geq \varepsilon,\ H_{n-5}(\mathbf{k}', \mathbf{p}_0) \geq \varepsilon, \ldots, H_w(\mathbf{k}', \mathbf{p}_0) \geq \varepsilon$$

2. Impose k^+_{lq+1} to be any computed upper bound of the $\max_{\mathbf{p}\in\mathcal{P}} \mathrm{tr}(P(\mathbf{k}', \mathbf{p}))$.

Remark 4. The stabilizing $\mathbf{k}' \in \mathcal{K}$ can be obtained by applying the same genetic/interval algorithm that is used to solve problem (31). Indeed, it suffices to omit k_{lq+1} and $\Phi(\lambda_{n+1}(\mathbf{k}_a))$ in the objective function. In this way we can also check the validity of Assumption 1. Computation of k^+_{lq+1} can be performed by any interval algorithm for global optimization over a given multidimensional interval (Hansen, 1992).

The overall problem is solved by means of a hybrid genetic/interval algorithm for global optimization developed by the authors (Guarino Lo Bianco and Piazzi, 2001a). More in detail, the finite minimization problem (36) is handled by a partially elitistic genetic algorithm which stochastically assures convergence to a global minimizer $\mathbf{k}^*_a$. while the necessary computation of the penalty terms $\Phi(\lambda_1(\mathbf{k}_a))$, $\Phi(\lambda_2(\mathbf{k}_a))$ and $\Phi(\lambda_{n+1}(\mathbf{k}_a))$ is obtained through an interval procedure. This procedure is a deterministic method converging with certainty thus the feasibility of the proposed minimizer is guaranteed. The interval procedure is described in Lo Bianco and Piazzi (2001a) while an improved version of can be found in Lo Bianco and Piazzi (2001b).

4. A TITO system application

In this example, problem (8) is solved for the following TITO system $\Sigma(\mathbf{p})$ ($n = 3$, $q = 2$, and $l = 2$):

$$A(\mathbf{p}) := \begin{bmatrix} p_1^2 & 0 & 1 \\ -p_1 p_2 & -2 & p_2 p_3 \\ 0 & p_3^2 & 0 \end{bmatrix} \quad B := \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \quad C := \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

where $\mathbf{p} := [p_1 \;\; p_2 \;\; p_3]^T \in \mathcal{P} := [1, 2] \times [1, 2] \times [3, 4]$ represents the plant parametric uncertainty vector ($m = 3$). The output-to-input feedback matrix is

$$K(\mathbf{k}) := \begin{bmatrix} k_1 & k_2 \\ k_3 & k_4 \end{bmatrix}$$

where $\mathbf{k} := [k_1 \;\; k_2 \;\; k_3 \;\; k_4]^T \in \mathcal{K} := [-100, 100] \times [-100, 100] \times [-100, 100] \times [-100, 100]$ is the design parameter vector ($\bar{k} = 100$). Finally define

$$Q := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 5 & 1 \\ 0 & 1 & 1 \end{bmatrix} \geq 0 \quad R := \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} > 0$$

the weighting matrixes being, respectively, positive semi-definite and definite. Both matrixes can be decomposed as $Q = D^T D$ and $R = L^T L$ with

$$D = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix} \quad L = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 1 \end{bmatrix}.$$

The genetic/interval algorithm used to solve (36) has been coded in C++ and adopts the interval operators of the PROFIL library (Knüppel, 1993).

By applying the genetic/interval algorithm to the semi-infinite inequalities of Step 1 (cf. the two-step procedure of Section 3) we have found the robust solution $\mathbf{k}' = [79.525 \;\; 63.868 \;\; -69.119 \;\; -96.880]^T$. This verifies the validity of Assumption 1. According with the second step of the preliminary procedure, a correct value of k_5^+ is 2000. Hence $\mathcal{K}_a := \mathcal{K} \times [0, 2000]$. By direct inspection of the observability matrix it is easy to verify that pair $(D, A(\mathbf{p}))$ is observable (hence, detectable) for all $\mathbf{p} \in \mathcal{P}$. This suffices for Property 3 to hold. On the other hand Assumption 2 still holds as it has been verified by using the genetic/interval algorithm in association with the characteristic polynomial of matrix $A(\mathbf{p}) + MD$.

Application of the genetic/interval algorithm to problem (36) has permitted finding an estimate of the global minimizer $\mathbf{k}^*$ given by the vector $[61.827 \;\; 26.570 \;\; -33.868 \;\; -25.819]^T$ with related estimate of the (worst-case) global minimum $J^* := \max_{\mathbf{p} \in \mathcal{P}} \{\mathrm{tr}(P(\mathbf{k}^*, \mathbf{p}))\}$ given by 1362.0. This is the better result obtained over 10 runs of the algorithm. Table 1 shows all the minimizers found together with their cost indexes and evaluation times (a PC Pentium IV 1.5 GHz has been used). It is worth nothing that the time burden due to the genetic algorithm is close to 4" while the remaining time is spent by the interval procedure to verify with certainty the feasibility of the solution. As desired, the genetic algorithm permits finding an estimate of the global minimizers introducing a moderated time burden.

By inspecting Table 1 it is possible to note that different minimizers have similar cost indexes (the worst cost index is the 5% greater than the best). Some tests have been executed in order to verify if this is due to a particularly "flat" cost function: an evident increase of the performance index has been detected moving away from any solution $\mathbf{k}^*$ found by the genetic/interval algorithm. This means that the genetic/interval algorithm has stopped in the neighborhoods of a cluster of minima.

The result obtained has been verified by simulating the closed-loop system with Simulink for $p_1 \in [1, 2]$, $p_2 \in [1, 2]$ and p_3 respectively equal to 3 and 4. The resulting shape of

Table 1. Results obtained over 10 runs of the genetic/interval algorithm.

J^*	Comp. time [s]	k_1^*	k_2^*	k_3^*	k_4^*
1407.8	6081	68.064	33.645	−38.687	−35.231
1362.4	4697	58.853	25.370	−31.156	−24.156
1384.3	7196	52.134	31.455	−25.380	−25.130
1361.9	6069	61.827	26.570	−33.868	−25.819
1399.1	6382	76.227	27.280	−48.048	−31.835
1420.5	5180	65.367	34.278	−36.264	−35.790
1399.4	6227	76.248	29.973	−48.253	−34.254
1432.6	4347	63.003	36.578	−35.302	−38.658
1406.2	6067	53.244	42.128	−25.018	−29.998
1371.8	5933	56.063	25.246	−29.738	−25.419

$\mathrm{tr}(P(\mathbf{k}^*, \mathbf{p}))$ is graphically shown in figure 1. The eigenvalues of the closed-loop system for the nominal plant, i.e. that associated to $\mathbf{p}_N := [1.5\ \ 1.5\ \ 3.5]^T$, are

$$\sigma(A(\mathbf{p}_N) + BK(\mathbf{k}^*)C) = \left\{ \begin{matrix} -3.5969 \\ -10.9861 + j10.1469 \\ -10.9861 - j10.1469 \end{matrix} \right\}.$$

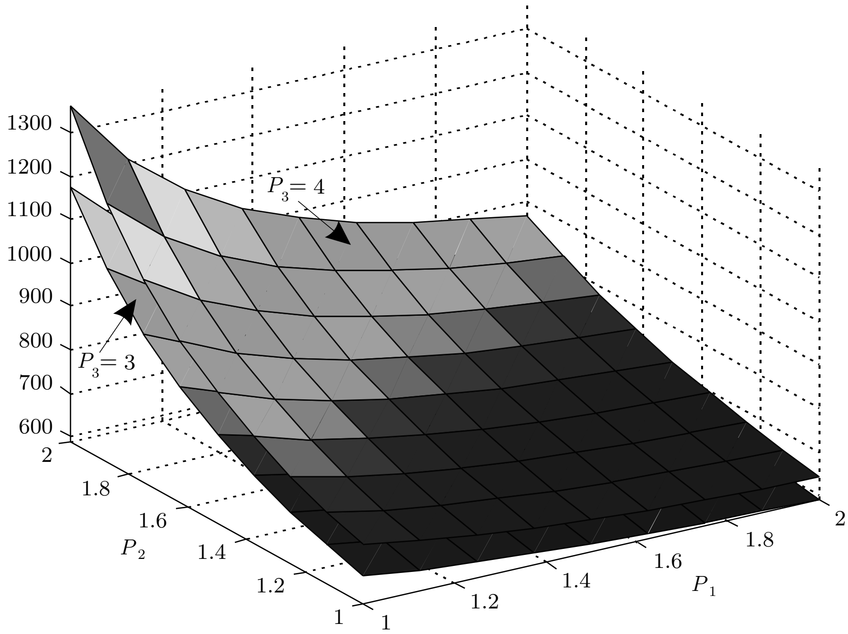

Figure 1. Optimal performance index $\mathrm{tr}(P(\mathbf{k}^*, \mathbf{p}))$.

By means of the interval procedure described in Piazzi and Marro (1996) it has been possible to establish that the minimum degree of stability is obtained for $\mathbf{p}_W := [1 \quad 2 \quad 4]^T$ where

$$\sigma(A(\mathbf{p}_W) + BK(\mathbf{k}^*)C) = \left\{ \begin{matrix} -2.5114 \\ -12.1539 + j14.8987 \\ -12.1539 - j14.8987 \end{matrix} \right\}.$$

5. Conclusions

In this paper, dealing with linear systems nonlinearly affected by uncertain parameters, we have presented a method for solving the worst-case H_2 optimal synthesis of static output feedbacks. What are the limitations and potentialities of the proposed semi-infinite optimization reformulation? From a computational complexity viewpoint, it stands clear that the posed worst-case H_2 problem, in its full generality, is NP-hard. Indeed, it encompasses as special cases both the standard static output feedback synthesis with no uncertainties and the robust stability analysis. The latter has been shown to be NP-hard (Nemirovskii, 1993; Poljak and Rohn, 1993) and the former may be NP-hard too (Syrmos et al., 1997). As a consequence, it is not realistic to solve large, and even moderate-size worst-case H_2 problems. Nevertheless, the numerical solution proposed via the hybrid genetic/interval algorithm (Guarino Lo Bianco and Piazzi, 2001a, 2001b) could lead to satisfactory solutions for a bunch of previously unsolved problems with control engineering pertinence.

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