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Robust set-point constrained regulation via dynamic inversion‡

Aurelio Piazzi*,†,1 and Antonio Visioli[2]

[1] *Dipartimento di Ingegneria dell'Informazione, University of Parma, Parco Area delle Scienze 181A, I-43100 Parma, Italy*
[2] *Dipartimento di Elettronica per l'Automazione, University of Brescia, Italy*

SUMMARY

In this paper we present a novel methodology, based on dynamic system inversion, for the set-point constrained regulation of a scalar plant in presence of structured uncertainties. The approach consists in choosing a polynomial as the output function and then designing both the controller and the reference input in order to minimize the worst-case settling-time of the set-point regulation, with constraints on the maximum absolute value of the control variable and on the maximum overshoot of the output. A comparison with a traditional methodology, which is based on the step-response, shows how the overall design is significantly simplified and how the worst-case settling-time can be greatly reduced. Optimization has been performed by means of genetic algorithms. Copyright © 2001 John Wiley & Sons, Ltd.

KEY WORDS: robust control; constrained regulation; dynamic inversion; optimization

1. INTRODUCTION

In spite of many efforts, the general problem of robust regulation of uncertain systems is a topic which still deserves intensive research investigation in order to obtain significant and applicable results. Indeed, in contrast with the important improvements on robust stability [1–3], the subject of robust performance, which is crucial for effective robust regulation, has fewer results, mainly devoted to robust performance analysis [4–7].

In this paper we reconsider, with a new perspective, the very basic problem of robust set-point regulation of minimum-phase scalar systems which are nonlinearly affected by uncertain parameters. Specifically, considering the problem of performing a robust 'transition' from a previous set-point value to a new one, under overshoot and actuator limitations, we propose

* Correspondence to: Aurelio Piazzi, Dipartimento di Ingegneria dell'Informazione, University of Parma, Parco Area delle Scienze 181A, I-43100 Parma, Italy.
† E-mail: aurelio@ce.unipr.it

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a new feedforward/feedback synthesis design with the aim of minimizing the worst-case settling-time relative to the transition. The design philosophy is to use feedback to reduce the sensitivity to plant parametric uncertainties, and then use dynamic system inversion, applied to the 'nominal' model, in order to obtain a reference input corresponding to an arbitrarily smooth output without oscillations. The final design step is done by solving a simplified optimization problem. Clearly, the main novelty lies in adopting a dynamic inversion procedure. Since 1960s system invertibility has been the subject of many investigations, especially for linear multivariable systems for which issues such as test conditions for system invertibility, algorithmic constructions of the inverse system and stability of the inverse system were addressed [8–11]. Another issue is left-invertibility or right-invertibility of a system. The latter concept is also known with a more illuminating term as functional reproducibility, which indicates the possibility to reproduce a given output function by a suitable input function [12]. In recent years, concepts borrowed from system invertibility have been applied to perfect or quasi-perfect tracking for linear and nonlinear dynamic systems [13–15]. Considering that the plant is scalar and minimum-phase, the functional reproducibility property necessary to perform a system noncausal inversion is always secured.

The paper is organized as follows. In Section 2 the robust constrained regulation problem is formally posed. The controller design procedure is presented in Section 3. Sections 4 and 5 describe the output synthesis and the dynamic inversion methodology, respectively. An optimization procedure, based on genetic algorithms, to deal with the uncertainties of the plant, is proposed in Section 6 while Section 7 reports a fixed-structure controller design based on the traditional unity-feedback scheme with a step-reference input. A design example shows, in Section 8, the definite improvement of the new approach over the traditional one. Conclusions are drawn in the last section.

The following notations are used in the paper. The sets of real positive and real negative numbers are denoted by $\mathbb{R}^+$ and $\mathbb{R}^-$, respectively. Given a real multidimensional interval $\mathcal{Q} \subseteq \mathbb{R}^l$, mid($\mathcal{Q}$) denotes the midpoint or center of $\mathcal{Q}$. The ith derivative of a real function $f(t)$ is designated with $f^{(i)}(t)$ or $D^i f(t)$. The $\mathbf{L}_\infty$ and $\mathbf{L}_1$ norms are defined as $\| f(\cdot) \|_\infty := \sup_{t \geqslant 0} | f(t) |$ and $\| f(\cdot) \|_1 := \int_0^\infty | f(t) | \, dt$. With $C^{(i)}$ we denote the class of functions which are continuous till to the ith derivative over $\mathbb{R}$. The unit-step function is indicated by $1(t)$, i.e. $1(t) = 0$ for $t < 0$ and $1(t) = 1$ for $t \geqslant 0$.

2. PROBLEM FORMULATION

In the context of linear, time-invariant, continuous-time systems, consider a scalar minimum-phase plant whose transfer function $P(s;\mathbf{q})$ is strictly proper, rational function nonlinearly depending on an uncertain parameter vector $\mathbf{q}$. However, it is known that $\mathbf{q} = [q_1, \dots, q_l]^{\mathrm{T}}$ belongs to a given multidimensional interval (box) $\mathcal{Q} = [q_1^-, q_1^+] \times \cdots \times [q_l^-, q_l^+]$. The structure of $P(s;\mathbf{q})$ is as follows ($h \in \{0, 1\}$):

$$P(s; \mathbf{q}) = \frac{b(s;\mathbf{q})}{s^h a(s;\mathbf{q})} \tag{1}$$

$$a(s; \mathbf{q}) = \sum_{i=0}^{n-h} a_i(\mathbf{q}) \, s^i \tag{2}$$

$$b(s; \mathbf{q}) = \sum_{i=0}^{m} b_i(\mathbf{q}) \, s^i \tag{3}$$

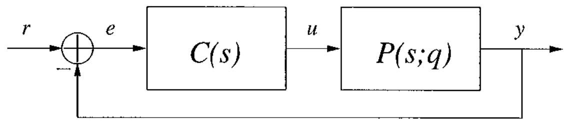

Figure 1. The unity-feedback system.

The order of the plant is n, its relative order is $\rho := n - m \geqslant 1$; coefficients $a_i(\mathbf{q})$ and $b_i(\mathbf{q})$ are continuous nonlinear functions over $\mathscr{Q}$. Both polynomials $a(s; \mathbf{q})$ and $b(s; \mathbf{q})$ are (Hurwitz) stable for all $\mathbf{q} \in \mathscr{Q}$. In order to avoid degeneracies, we assume

$$a_{n-h}(\mathbf{q}) > 0 \quad \forall \mathbf{q} \in \mathscr{Q} \tag{4}$$

$$\{b_m(\mathbf{q}) > 0 \forall \mathbf{q} \in \mathscr{Q}\} \vee \{b_m(\mathbf{q}) < 0 \ \forall \mathbf{q} \in \mathscr{Q}\} \tag{5}$$

Consider now the unity-feedback system shown in Figure 1. The aim of the overall feedforward/feedback control is to obtain a 'robust' transition from a previous set-point value y_0 to a new one y_1. Without loss of generality, we choose $y_0 = 0$ and $y_1 > 0$. The first requirement is the robust stability of the closed-loop over the uncertain domain $\mathscr{Q}$. Moreover, this transition has to satisfy an overshoot limitation, a saturation constraint on $u(t)$ and has to minimize the (worst-case) settling time. Simply stated, the posed problem is: determine a reference function $r(t)$ and a controller $C(s)$ such that

1. the closed-loop system is stable for all $\mathbf{q} \in \mathscr{Q}$;
2. $\lim_{t \to \infty} y(t) = y_1$ for all $\mathbf{q} \in \mathscr{Q}$ (steady-state condition);
3. the overshoot in response to $r(t)$ is bounded by a given $\bar{S}$ for all $\mathbf{q} \in \mathscr{Q}$;
4. the absolute value of the manipulative input $u(t)$ is bounded by a given u_{sat} for all $\mathbf{q} \in \mathscr{Q}$;
5. it is minimized the worst-case settling time.

In full generality the above problem is an extraordinarily difficult problem. In the following sections, we devise a viable sub-optimal solution by means of a four-step design procedure.

3. CONTROLLER DESIGN

Introduce the 'nominal' parameter vector $\mathbf{q}^0 := \text{mid}(\mathscr{Q})$. Then choose the following parameterized controller, which is based on nominal pole-zero cancellations:

$$C(s; \alpha) := \alpha \frac{a(s; \mathbf{q}^0)}{sb(s; \mathbf{q}^0) d(s)} \quad \text{if } h = 0$$

$$C(s; \alpha) := \alpha \frac{a(s; \mathbf{q}^0)}{b(s; \mathbf{q}^0) d(s)} \quad \text{if } h = 1$$

The introduced $d(s)$ is a user-chosen stable monic polynomial of order $\rho - 1$; if $\rho = 1$ then $d(s) := 1$. The resulting controller is biproper and its order is n if $h = 0$ or $n - 1$ if $h = 1$. In both cases the characteristic polynomial of the closed-loop system is given by

$$sd(s)b(s; \mathbf{q}^0)\, a(s; \mathbf{q}) + \alpha a(s; \mathbf{q}^0)\, b(s; \mathbf{q}) \tag{6}$$

Property 1

There exists a sufficiently small $\alpha \in \mathbb{R}^+$ such that the closed-loop system is stable for all $\mathbf{q} \in \mathcal{Q}$.

Proof. Without loss of generality, we consider $h = 0$ so that the degree of the characteristic polynomial (6) is equal to $2n$ (this degree is $n - 1$ if $h = 1$). Denote with $z_i(\mathbf{q}, \alpha)$ $i = 1, \ldots, 2n$, the roots of (6), i.e. the closed-loop poles of the feedback system. The open-loop poles are then given by $z_i(\mathbf{q}, 0)$ $i = 1, \ldots, 2n$, and we can assume that $z_1(\mathbf{q}, 0) = 0$. The velocity constant of the feedback system is

$$K_v = \alpha \frac{a(0; \mathbf{q}^0)\, b(0; \mathbf{q})}{d(0) b(0; \mathbf{q}^0)\, a(0; \mathbf{q})} = \alpha \frac{a_0(\mathbf{q}^0)\, b_0(\mathbf{q})}{d(0) b_0(\mathbf{q}^0)\, a_0(\mathbf{q})} \tag{7}$$

and it is always positive for all $\mathbf{q} \in \mathcal{Q}$ and $\alpha \in \mathbb{R}^+$. As a consequence, by a well-known property of the root locus technique, the root $z_1(\mathbf{q}, \alpha) \in \mathbb{R}^-$ for a sufficiently small $\alpha \in \mathbb{R}^+$. Formally, there exists $\alpha_1 \in \mathbb{R}^+$ such that

$$z_1(\mathbf{q}, \alpha) < 0 \quad \forall \mathbf{q} \in \mathcal{Q} \quad \text{and} \quad \forall \alpha \in (0, \alpha_1] \tag{8}$$

By virtue of the introduced assumptions, the open-loop poles $z_i(\mathbf{q}, 0)$ $i = 2, \ldots, 2n$ are stable (belong to the open left half-plane) for all $\mathbf{q} \in \mathcal{Q}$. Hence, having defined

$$\sigma := \max_{\mathbf{q} \in \mathcal{Q}} \max_{i = 2, \ldots, 2n} \{\mathrm{Re}(z_1(\mathbf{q}, 0))\} \tag{9}$$

it follows that $\sigma < 0$.

The characteristic polynomial (6) is a continuous function of the arguments $\mathbf{q} \in \mathcal{Q}$ and $\alpha \in \mathbb{R}$. Therefore, the following limits hold uniformly over $\mathcal{Q}$:

$$\lim_{\alpha \to 0^+} [z_i(\mathbf{q}, \alpha) - z_i(\mathbf{q}, 0)] = 0, \quad i = 2, \ldots, 2n$$

Hence, there exists $\alpha_i > 0$ such that $(i = 2, \ldots, 2n)$

$$|z_i(\mathbf{q}, \alpha) - z_i(\mathbf{q}, 0)| < |\sigma| \ \forall \mathbf{q} \in \mathcal{Q}, \quad \forall \alpha \in (0, \alpha_i] \tag{10}$$

Define

$$\alpha' := \min_{i = 1, \ldots, 2n} \{\alpha_i\}$$

and from (10) it follows that $(i = 2, \ldots, 2n)$

$$|z_i(\mathbf{q}, \alpha') - z_i(\mathbf{q}, 0)| < |\sigma| \quad \forall \mathbf{q} \in \mathcal{Q}$$

Considering definition (9) we then infer

$$\mathrm{Re}(z_i(\mathbf{q}, \alpha')) < 0 \ \forall \mathbf{q} \in \mathcal{Q}, \quad i = 2, \ldots, 2n \tag{11}$$

From (8), obtain eventually

$$\mathrm{Re}(z_1(\mathbf{q}, \alpha')) = z_1(\mathbf{q}, \alpha') < 0 \quad \forall \mathbf{q} \in \mathcal{Q} \tag{12}$$

Choose $\alpha := \alpha'$ and statements (11) and (12) prove the robust stability of the closed-loop system.

4. OUTPUT SYNTHESIS

Determine a function of class $C^{(p)}$ in order to permit an arbitrarily smooth transition from 0 to y_1, to be completed without oscillations in the interval $[0, \tau]$, where τ is a positive real parameter.

Specifically, consider over the domain $[0, \tau]$ a polynomial $y(t)$ of order $2p+1$:

$$y(t) = c_{2p+1} t^{2p+1} + c_{2p} t^{2p} + \cdots + c_1 t + c_0$$

Determine the $2p+2$ coefficients by solving the following parameterized system with $2p+2$ linear equations:

$$\begin{aligned}
&y(0) = 0, \quad y(\tau) = y_1 \\
&y^{(1)}(0) = 0, \quad y^{(1)}(\tau) = 0 \\
&\vdots \\
&y^{(p)}(0) = 0, \quad y^{(p)}(\tau) = 0
\end{aligned}$$

The above algebraic system always admits a unique solution which is given by the following closed-form expression:

$$y(t; \tau) = y_1 \beta(p, \tau) \int_0^t v^p (\tau - v)^p \, \mathrm{d}v \tag{13}$$

The positive coefficient $\beta(p, \tau)$ is given by [17]

$$\beta(p, \tau) := \left(\int_0^\tau v^p (\tau - v)^p \, \mathrm{d}v \right)^{-1} = \frac{(2p+1)!}{(p!)^2 \tau^{2p+1}} \tag{14}$$

Derivation of (13) is easy by considering that

$$y^{(1)}(t; \tau) = y_1 \beta(p, \tau) t^p (\tau - t)^p$$

Moreover, $y^{(1)}(t; \tau)$ is always strictly positive in the open interval $(0, \tau)$ so that $y(t; \tau)$ does not exhibit swing modes. Outside the interval $[0, \tau]$ the function $y(t; \tau)$ is equal to 0 for $t < 0$ and equal to y_1 for $t > \tau$. A fundamental property of the synthesized 'transition' function signal $y(t; \tau)$ is the following.

Property 2

Let be given an arbitrary (Hurwitz) stable scalar system with transfer function

$$H(s) = \frac{\delta_m s^m + \delta_{m-1} s^{m-1} + \cdots + \delta_0}{\beta_m s^m + \beta_{m-1} s^{m-1} + \cdots + \beta_0} \tag{15}$$

The input of this system be $y(t; \tau)$ and the corresponding output, with zero initial conditions at $t = 0$, be denoted by $z(t; \tau)$. Then, it holds

$$\lim_{\tau \to \infty} \| z(\cdot; \tau) - H(0) y(\cdot; \tau) \|_\infty = 0 \tag{16}$$

Proof. Introduce the Laplace transforms $Z(s;\tau) := \mathscr{L}[z(t;\tau)]$ and $Y(s;\tau) := \mathscr{L}[y(t;\tau)]$; hence, $Z(s;\tau) = H(s)\,Y(s;\tau)$. It follows that

$$
\begin{aligned}
Z(s;\tau) - H(0)\,Y(s;\tau) &= [H(s) - H(0)]\,Y(s;\tau) \\
&= \frac{(\beta_0\delta_m - \delta_0)\,s^m + (\beta_0\delta_{m-1} - \delta_0\beta_{m-1})\,s^{m-1} + \cdots + (\beta_0\delta_1 - \delta_0\beta_1)\,s}{\beta_0\,(s^m + \beta_{m-1}\,s^{m-1} + \cdots + \beta_0)}\; Y(s;\tau) \\
&= \frac{(\beta_0\delta_m - \delta_0)\,s^{m-1} + \cdots + (\beta_0\delta_1 - \delta_0\beta_1)}{\beta_0(s^m + \beta_{m-1}s^{m-1} + \cdots + \beta_0)}\; sY(s;\tau) \\
&=: \tilde{H}(s)\,sY(s;\tau)
\end{aligned}
$$

Note that $\mathscr{L}[Dy(t;\tau)] = sY(s;\tau)$ because $y(t;\tau)$ is a continuous function over $\mathbb{R}$ (for all integers $p \geqslant 0$) with $y(0;\tau) = 0$. As a consequence, the signal $z(t;\tau) - H(0)\,y(t;\tau)$ can be considered as the output of the (Hurwitz) stable system $\tilde{H}(s)$ with input $Dy(t;\tau)$. Taking into account (see (13) and (14))

$$
Dy(t;\tau) = \begin{cases} 0 & \text{if } t \in (-\infty,\ 0) \\ y_1\,\frac{(2p+1)!}{(p!)^2\tau^{2p+1}}\,t^p(\tau - t)^p & \text{if } t \in [0,\ \tau] \\ 0 & \text{if } t \in (\tau,\ +\infty) \end{cases}
$$

we deduce easily that

$$
\|Dy(\cdot;\tau)\|_\infty = y_1\,\frac{(2p+1)!}{(p!)^2\,\tau^{2p+1}}\left(\frac{1}{2}\,\tau\right)^{2p} = y_1\,\frac{(2p+1)!}{(p!)^2\,2^{2p}\,\tau} \tag{17}
$$

Denoting with $\tilde{h}(t) := \mathscr{L}^{-1}[\tilde{H}(s)]$ it follows that

$$
\|z(\cdot;\tau) - H(0)\,y(\cdot;\tau)\|_\infty \leqslant \int_0^\infty |\tilde{h}(t)|\,\mathrm{d}t\,\|Dy(t;\tau)\|_\infty \tag{18}
$$

The integral appearing in (18) is the peak gain of $\tilde{H}(s)$ and it is finite because $\tilde{H}(s)$ is (Hurwitz) stable [16]. Therefore, limit (16) follows from inequality (18) and relation (17). □

Property 3

The global maximum of $|D^i y(t;\tau)|$ over $\mathbb{R}$ can be expressed as $(i = 0, 1, \ldots, p)$

$$
\max_{t\in\mathbb{R}} |D^i\,y(t;\tau)| = c_{pi}\,\frac{y_1}{\tau^i} \tag{19}
$$

where c_{pi} is a positive constant that does not depend on τ.

Proof. See Reference [17]. □

5. DYNAMIC INVERSION

At this stage of the design procedure, considering the nominal closed-loop system, we synthesize the reference input $r(t;\alpha,\tau)$ corresponding to the desired regulated output given by the 'transition' function $y(t; \tau)$. The method to obtain $r(t; \alpha, \tau)$ is based on dynamic system inversion. The nominal open-loop transfer function is $\alpha/(sd(s))$ and the corresponding transfer function between r and y is (see Section 3)

$$T_{yr}(s; \alpha) = \frac{\alpha}{sd(s) + \alpha} \tag{20}$$

where $d(s)$ can be defined according to:

$$d(s) := s^{\rho-1} + d_{\rho-2}s^{\rho-2} + \cdots + d_0 \tag{21}$$

Denote by $R(s; \alpha, \tau)$ and $Y(s; \tau)$ the Laplace transforms of $r(t; \alpha, \tau)$ and $y(t; \tau)$ respectively. Considering that both $r(t; \alpha, \tau) = 0$ and $y(t; \tau) = 0$ for all $t < 0$ (at $t = 0^-$ all initial conditions are zeros), it follows that

$$Y(s; \tau) = T_{yr}(s; \alpha)\, R(s; \alpha, \tau)$$

The required system inversion can be simply performed by computing $T_{yr}(s; \alpha)^{-1}$:

$$R(s; \alpha, \tau) = T_{yr}(s; \alpha)^{-1}\, Y(s; \tau)$$

From (20) and (21) we obtain:

$$T_{yr}(s; \alpha)^{-1} = \frac{1}{\alpha}s^{\rho} + \frac{d_{\rho-2}}{\alpha}s^{\rho-1} + \cdots + \frac{d_0}{\alpha}s + 1 \tag{22}$$

Therefore, it holds

$$R(s; \alpha, \tau) = \frac{1}{\alpha}s^{\rho}\,Y(s; \tau) + \frac{d_{\rho-2}}{\alpha}s^{\rho-1}\,Y(s; \tau) + \cdots + \frac{d_0}{\alpha}sY(s; \tau) + Y(s; \tau)$$

Taking the inverse Laplace transform of the above relation, we finally derive the closed-form expression of the sought reference input:

$$r(t; \alpha, \tau) = \frac{1}{\alpha}D^{\rho}y(t; \tau) + \frac{d_{\rho-2}}{\alpha}D^{\rho-1}\,y(t; \tau) + \cdots + \frac{d_0}{\alpha}Dy(t; \tau) + y(t; \tau) \tag{23}$$

An alternative expression of $r(t; \alpha, \tau)$ is

$$r(t; \alpha, \tau) = y(t; \tau) + \alpha^{-1}\sum_{i=1}^{\rho} d_{i-1}D^{i}\,y(t; \tau) \tag{24}$$

having defined $d_{\rho-1} = 1$.

Remark 1

It is worth stressing that expression (24) is simply formed by an algebraic linear combination of $y(t; \tau)$ and its derivatives (till to the ρth order) without any inclusion of zero dynamics modes. This is due to the absence of finite zeros in $T_{yr}(s; \alpha)$. Omitting the parameter arguments in expression (24), this can be interpreted as the differential equation describing the system behaviour of the nominal closed-loop system.

Remark 2

The explicit expression (24) also makes apparent that the desired output $y(t;\tau)$ must belong to $C^{(p)}$ in order to obtain a continuous reference signal. This requires that p (cf. Section 4) be greater or equal to the relative order ρ of the controlled plant. In general, we have that $r(t;\alpha,\tau)\in C^{(p-\rho)}$ if and only if $y(t;\tau)\in C^{(p)}$.

6. OPTIMIZATION FOR ROBUSTNESS

The previous Sections 3–5 depict a noncausal regulation scheme that depends on the parameters α and τ. In order to achieve a worst-case optimal robust performance, we strive to optimally choose an α value, which pertains to the controller and more specifically to the velocity constant of the feedback system, and a τ value which is directly tied to the expected settling-time of the set-point regulation. First consider the robust stability issue. Denote by $\xi(s;\alpha,\mathbf{q}):=\sum_{i=0}^{n_c}\xi_i(\alpha,\mathbf{q})\, s^i$ ($n_c := 2n$ if $h=0$ and $n_c := 2n-1$ if $h=1$) the characteristic polynomial defined as follows (cf. (6)):

$$\xi(s;\alpha,\mathbf{q}) := (-1)^e\,[sd(s)\,b(s;\mathbf{q}^0)\,a(s;\mathbf{q}) + \alpha a(s;\mathbf{q}^0)\,b(s;\mathbf{q})]$$

where $e=2$ if $b_m(q^0)>0$ and $e=1$ if $b_m(\mathbf{q}^0)<0$ (cf. (5)). The ith-order Hurwitz determinant associated to $\xi(s;\alpha,\mathbf{q})$ is denoted by $H_i(\alpha,\mathbf{q})$. The following result can be easily established by virtue of the Lienard and Chipard criterion [18].

Property 4

The closed-loop system is stable for all $\mathbf{q}\in\mathcal{Q}$ if and only if the following inequalities hold, for all $\mathbf{q}\in\mathcal{Q}$:

$$\xi_0(\alpha,\mathbf{q})>0;\ \xi_1(\alpha,\mathbf{q})>0,\ \xi_3(\alpha,\mathbf{q})>0,\ \dots,\ \xi_v(\alpha,\mathbf{q})>0;$$

$$H_{n_c-1}(\alpha,\mathbf{q})>0,\ H_{n_c-3}(\alpha,\mathbf{q})>0,\ \dots,\ H_w(\alpha,\mathbf{q})>0.$$

with $v := n_c-1$, $w := 3$ if n_c is even, while $v := n_c-2$, $w := 2$ if n_c is odd.

The reference input $r(t;\alpha,\tau)$ reported in (24) be applied to the uncertain unity-feedback system and denote, respectively, by $u_a(t;\alpha,\tau,\mathbf{q})$ and $y_a(t;\alpha,\tau,\mathbf{q})$ the corresponding input and output of the plant. Note that, by construction, $y_a(t;\alpha,\tau,\mathbf{q}^0)\equiv y(t;\tau)$. Considering that the closed-loop is of type 1, it follows that the steady-state requirement is always satisfied provided that the robust stability is secured:

$$\lim_{t\to\infty} y_a(t;\alpha,\tau,\mathbf{q}) = y_1 \quad \forall\tau\in\mathbb{R}^+ \quad \text{and} \quad \forall\mathbf{q}\in\mathcal{Q}$$

Let us define the settling time as the minimum time after that the regulated output remains within a 2 per cent range of y_1:

$$t_s(\alpha,\tau,\mathbf{q}) := \min\{s\in\mathbb{R}^+ : |y_a(t;\alpha,\tau,\mathbf{q})-y_1|\leqslant 0.02y_1\ \forall t\geqslant s\}$$

Then, the following semi-infinite minimax optimization problem emerges:

$$\min_{\alpha,\,\tau\in\mathbb{R}^+}\ \max_{\mathbf{q}\in\mathbb{Q}}\ \{t_s(\alpha,\tau,\mathbf{q})\} \tag{25}$$

subject to, for all $\mathbf{q} \in \mathcal{Q}$,

$$\xi_0(\alpha, \mathbf{q}) \geqslant \varepsilon; \quad \xi_1(\alpha, \mathbf{q}) \geqslant \varepsilon, \quad \xi_3(\alpha, \mathbf{q}) \geqslant \varepsilon, \ldots, \xi_v(\alpha, \mathbf{q}) \geqslant \varepsilon \tag{26}$$

$$H_{n_c-1}(\alpha, \mathbf{q}) \geqslant \varepsilon, \quad H_{n_c-3}(\alpha, \mathbf{q}) \geqslant \varepsilon, \ldots, H_w(\alpha, \mathbf{q}) \geqslant \varepsilon \tag{27}$$

$$y_a(t; \alpha, \tau, \mathbf{q}) \leqslant (1 + 0.01\bar{S})\, y_1 \quad \forall t \geqslant 0 \tag{28}$$

$$|u_a(t; \alpha, \tau, \mathbf{q})| \leqslant u_{\text{sat}} \quad \forall t \geqslant 0 \tag{29}$$

where the constant ε is a sufficiently small positive value.

Theorem 1
Let be given any $\bar{S} > 0$ and any u_{sat} satisfying

$$u_{\text{sat}} > \max_{\mathbf{q} \in \mathcal{Q}} \left| \frac{a_0(\mathbf{q})}{b_0(\mathbf{q})} y_1 \right| \quad \text{if } h = 0, \tag{30}$$

$$u_{\text{sat}} > 0 \quad \text{if } h = 1. \tag{31}$$

Then the minimax problem (25) *always admits a solution.*

Proof. Property 1 assures the existence of a positive α for which robust closed-loop stability is secured. Hence, by virtue of Property 3, this α satisfies all the Lienard–Chipard constraints (26) and (27) for a sufficiently small $\varepsilon \in \mathbb{R}^+$. Introduce the transfer functions

$$G_{ur}(s; \alpha, \mathbf{q}) := \frac{C(s;\ \alpha)}{1 + C(s;\ \alpha)\ P(s;\ \mathbf{q})} \tag{32}$$

$$G_{yr}(s; \alpha, \mathbf{q}) := \frac{C(s;\ \alpha)\, P(s;\ \mathbf{q})}{1 + C(s;\ \alpha)\ P(s;\ \mathbf{q})} \tag{33}$$

Note that $G_{yr}(0; \alpha, \mathbf{q}) = 1\ \forall \mathbf{q} \in \mathcal{Q}$. By linear superposition, the signal $y_a(t; \alpha, \tau, \mathbf{q})$ can be rewritten as

$$y_a(t; \alpha, \tau, \mathbf{q}) = y_{a0}(t; \alpha, \tau, \mathbf{q}) + y_{a1}(t; \alpha, \tau, \mathbf{q}) \tag{34}$$

where $y_{a0}(t; \alpha, \tau, \mathbf{q})$ is the response output to reference input $y(t; \tau)$ and $y_{a1}(t; \alpha, \tau, \mathbf{q})$ is the response output to $\alpha^{-1} \sum_{i=1}^{\rho} d_{i-1} D^i y(t; \tau)$ (see (24)). Since the plant $P(s; \mathbf{q})$ depends continuously on $\mathbf{q}$, this implies that both signals $y_{a0}(t; \alpha, \tau, \mathbf{q})$ and $y_{a1}(t; \alpha, \tau, \mathbf{q})$ are continuous functions of $\mathbf{q}$ over $\mathcal{Q}$. By virtue of Property 2, we then have uniformly over $\mathcal{Q}$

$$\lim_{\tau \to \infty} \| y_{a0}(\cdot; \alpha, \tau, \mathbf{q}) - y(\cdot; \tau) \|_\infty = 0 \quad \forall \mathbf{q} \in \mathcal{Q}$$

Hence, for any given small $v \in \mathbb{R}^+$ there exists $\tau_1 \in \mathbb{R}^+$ such that

$$\| y_{a0}(\cdot;\ \alpha, \tau, \mathbf{q}) - y(\cdot;\ \tau) \|_\infty \leqslant v\ \forall \tau \in [\tau_1,\ \infty), \quad \forall \mathbf{q} \in \mathcal{Q} \tag{35}$$

As known, the peak gain of $G_{yr}(s; \alpha, \mathbf{q})$ is $\| g_{yr}(\cdot; \alpha, \mathbf{q}) \|_1$ where $g_{yr}(\cdot; \alpha, \mathbf{q})$ is the associated impulse response. As a consequence (cf. Property 3)

$$\begin{aligned} \| y_{a1}(\cdot; \alpha, \tau, \mathbf{q}) - y(\cdot; \tau) \|_\infty &\leqslant \| g_{yr}(\cdot; \alpha, \mathbf{q}) \|_1 \, \| \alpha^{-1} \sum_{i=1}^{\rho} d_{i-1} \, D^i y(t; \tau) \| \\ &\leqslant \| g_{yr}(\cdot; \alpha, \mathbf{q}) \|_1 \, \alpha^{-1} \sum_{i=1}^{\rho} d_{i-1} \, \| D^i y(t; \tau) \| \\ &= \| g_{yr}(\cdot; \alpha, \mathbf{q}) \|_1 \, \alpha^{-1} \sum_{i=1}^{\rho} d_{i-1} \, c_{pi} \tfrac{y_1}{\tau^i}. \end{aligned}$$

Then, the stability of $G_{yr}(s; \alpha, \mathbf{q})$ over $\mathcal{Q}$ implies that, for any given $v \in \mathbb{R}^+$, there exists $\tau_2 \in \mathbb{R}^+$ satisfying

$$\| y_{a1}(\cdot; \alpha, \tau, \mathbf{q}) \|_\infty \leqslant v \quad \forall \tau \in [\tau_2, \infty), \quad \forall \mathbf{q} \in \mathcal{Q} \tag{36}$$

Define $\tau_{12} := \max[\tau_1, \tau_2]$ and set $v := 0.005\bar{S}y_1$. From (35) and (36) we deduce

$$y_{a0}(t; \alpha, \tau, \mathbf{q}) \leqslant 0.005\bar{S}y_1$$

$$| y_{a1}(t; \alpha, \tau, \mathbf{q}) | \leqslant 0.005\bar{S}y_1$$

for all $t \geqslant 0$, $\tau \in [\tau_{12}, \infty)$, and $\mathbf{q} \in \mathcal{Q}$. These inequalities, taking into account (34), imply

$$y_a(t; \alpha, \tau, \mathbf{q}) \leqslant (1 + 0.01\bar{S}) y_1 \quad \forall t \geqslant 0, \quad \forall \tau \in [\tau_{12}, \infty), \quad \forall \mathbf{q} \in \mathcal{Q} \tag{37}$$

Inequality (37) proves the feasibility of constraint (28).
Now we focus our attention on the last constraint (29). We can again use the linear superposition to express the plant control input as

$$u_a(t; \alpha, \tau, \mathbf{q}) = u_{a0}(t; \alpha, \tau, \mathbf{q}) + u_{a1}(t; \alpha, \tau, \mathbf{q}) \tag{38}$$

where $u_{a0}(t; \alpha, \tau, \mathbf{q})$ and $u_{a1}(t; \alpha, \tau, \mathbf{q})$ are the outputs of $G_{ur}(s; \alpha, \mathbf{q})$ to the inputs $y(t; \tau)$ and $\alpha^{-1} \sum_{i=1}^{\rho} d_{i-1} D^i y(t; \tau)$, respectively. Following a reasoning similar to the previous one, we derive

$$\lim_{\tau \to \infty} \| u_{a0}(\cdot; \alpha, \tau, \mathbf{q}) - G_{ur}(0; \alpha, \mathbf{q}) \, y(\cdot; \tau) \|_\infty = 0 \quad \forall \mathbf{q} \in \mathcal{Q} \tag{39}$$

Examining the case $h = 0$, from (32) it follows that

$$G_{ur}(0; \alpha, \mathbf{q}) = \frac{a_0(\mathbf{q})}{b_0(\mathbf{q})} \tag{40}$$

For any given $\eta \in \mathbb{R}^+$ there exists $\tau_3 \in \mathbb{R}^+$ such that

$$\left\| u_{a0}(\cdot; \alpha, \tau, \mathbf{q}) - \frac{a_0(\mathbf{q})}{b_0(\mathbf{q})} \, y(\cdot; \tau) \right\|_\infty \leqslant \eta \quad \forall \tau \in [\tau_3, \infty), \quad \forall \mathbf{q} \in \mathcal{Q} \tag{41}$$

Denote with $g_{ur}(t; \alpha, \mathbf{q})$ the impulse response of $G_{ur}(s; \alpha, \mathbf{q})$ so that

$$\| u_{a1}(\cdot; \alpha, \tau, \mathbf{q}) \|_\infty \leqslant \| g_{ur}(\cdot; \alpha, \mathbf{q}) \|_1 \, \| \alpha^{-1} \sum_{i=1}^{\rho} d_{i-1} D^i y(\cdot; \tau) \|_\infty$$

$$\leqslant \| g_{ur}(\cdot; \alpha, \mathbf{q}) \|_1 \, \alpha^{-1} \sum_{i=1}^{\rho} d_{i-1} \, c_{pi} \, \tfrac{y_1}{\tau^i}.$$

From the above inequalities we infer that, for any given $\eta \in \mathbb{R}^+$, there exists $\tau_4 \in \mathbb{R}^+$ satisfying

$$\| u_{a1}(\cdot; \alpha, \tau, \mathbf{q}) \|_\infty \leqslant \eta \quad \forall \tau \in [\tau_4, \infty), \quad \forall \mathbf{q} \in \mathcal{Q} \tag{42}$$

Inequality (41) is equivalent to

$$\left| u_{a0}(t; \alpha, \tau, \mathbf{q}) - \frac{a_0(\mathbf{q})}{b_0(\mathbf{q})} y(t, \tau) \right| \leqslant \eta \quad \forall t \geqslant 0, \quad \forall \tau \in [\tau_3, \infty), \quad \forall \mathbf{q} \in \mathcal{Q}$$

Therefore,

$$| u_{a0}(t; \alpha, \tau, \mathbf{q}) | \leqslant \left| \frac{a_0(\mathbf{q})}{b_0(\mathbf{q})} y(t, \tau) \right| + \eta \quad \forall t \geqslant 0, \quad \forall \tau \in [\tau_3, \infty), \quad \forall \mathbf{q} \in \mathcal{Q}$$

Considering that the image of $y(t; \tau)$ over $\mathbb{R}$ is $[0, y_1]$ we deduce

$$| u_{a0}(t; \alpha, \tau, \mathbf{q}) | \leqslant \max_{\mathbf{q} \in \mathcal{Q}} \left| \frac{a_0(\mathbf{q})}{b_0(\mathbf{q})} y_1 \right| + \eta \quad \forall t \geqslant 0, \quad \forall \tau \in [\tau_3, \infty), \quad \forall \mathbf{q} \in \mathcal{Q} \tag{43}$$

On the other hand, from (42) it holds

$$| u_{a1}(t; \alpha, \tau, \mathbf{q}) | \leqslant \eta \quad \forall t \geqslant 0, \quad \forall r \in [\tau_4, \infty), \quad \forall \mathbf{q} \in \mathcal{Q} \tag{44}$$

Define $\tau_{34} := \max[\tau_3, \tau_4]$ and from (43) and (44) we obtain

$$| u_a(t; \alpha, \tau, \mathbf{q}) | \leqslant \max_{\mathbf{q} \in \mathcal{Q}} \left| \frac{a_0(\mathbf{q})}{b_0(\mathbf{q})} y_1 \right| + 2\eta \quad \forall t \geqslant 0, \quad \forall \tau \in [\tau_{34}, \infty), \quad \forall \mathbf{q} \in \mathcal{Q} \tag{45}$$

Taking into account (30) we choose

$$\eta := \frac{1}{2} \left(u_{\text{sat}} - \max_{q \in \mathcal{Q}} \left| \frac{a_0(\mathbf{q})}{b_0(\mathbf{q})} y_1 \right| \right)$$

so that (45) becomes

$$| u_a(t; \alpha, \tau, \mathbf{q}) | \leqslant u_{\text{sat}} \quad \forall t \geqslant 0, \quad \forall \tau \in [\tau_{34}, \infty), \quad \forall \mathbf{q} \in \mathcal{Q} \tag{46}$$

As a last step define $\tau_{14} := \max[\tau_{12}, \tau_{34}]$ and from (37) and (46) constraints (28) and (29) are satisfied for all $\tau \in [\tau_{14}, \infty)$. Hence, all the constraints are satisfied and this proves that problem (25) has a solution for the case $h = 0$. If $h = 1$, the first part of the proof holds till statement (39). Then, relation (40) is substituted with

$$G_{ur}(0; \alpha, \mathbf{q}) = 0 \quad \forall \mathbf{q} \in \mathcal{Q}$$

As a consequence, the subsequent reasoning runs, with simplifications, analogously and eventually we again conclude that all the constraints can be made feasible so that problem (25) admits a solution. □

In general, (25) is a difficult nonlinear minimax problem. To solve it, a possible way is to adopt the approximating techniques to Polak [19]. Also useful, with pertinent modifications, it can be the genetic/interval approach exposed in Reference [20]. In same cases, a useful (suboptimal) solution can be obtained by relaxing problem (25). Specifically, $\mathcal{Q}$ can be replaced with the finite set of its vertices $\{q^1, q^2, \ldots, q^{n_1}\}$ with $n_l := 2^l$: The relaxed problem is then

$$\min_{\alpha, \tau \in \mathbb{R}^+} \max \{t_s(\alpha, \tau, \mathbf{q}^i)\colon i = 1, \ldots, n_l\} \tag{47}$$

subject to $(i = 1, \ldots, n_l)$

$$\xi_0(\alpha, \mathbf{q}^i) \geqslant \varepsilon; \quad \xi_1(\alpha, \mathbf{q}^i) \geqslant \varepsilon, \quad \xi_3(\alpha, \mathbf{q}^i) \geqslant \varepsilon, \ldots, \xi_v(\alpha, \mathbf{q}^i) \geqslant \varepsilon$$

$$H_{n_c-1}(\alpha, \mathbf{q}^i) \geqslant \varepsilon, \quad H_{n_c-3}(\alpha, \mathbf{q}^i) \geqslant \varepsilon, \ldots, H_w(\alpha, \mathbf{q}) \geqslant \varepsilon$$

$$y_a(t; \alpha, \tau, \mathbf{q}^i) \leqslant (1 + 0.01\bar{S})\, y_1 \quad \forall t \geqslant 0$$

$$|u_a(t; \alpha, \tau, \mathbf{q}^i)| \leqslant u_{\text{sat}} \quad \forall t \geqslant 0$$

By introducing the slack variable $\eta \in \mathbb{R}^+$, problem (47) is equivalent to the following finite optimization problem:

$$\min_{\alpha,\ \tau,\ \eta \in \mathbb{R}^+} \{\eta\} \tag{48}$$

subject to $(i = 1, \ldots, n_l)$

$$\xi_0(\alpha, \mathbf{q}^i) \geqslant \varepsilon; \quad \xi_1(\alpha, \mathbf{q}^i) \geqslant \varepsilon, \quad \xi_3(\alpha, \mathbf{q}^i) \geqslant \varepsilon, \ldots, \xi_v(\alpha, \mathbf{q}^i) \geqslant \varepsilon$$

$$H_{n_c-1}(\alpha, \mathbf{q}^i) \geqslant \varepsilon, \quad H_{n_c-3}(\alpha, \mathbf{q}^i) \geqslant \varepsilon, \ldots, H_w(\alpha, \mathbf{q}) \geqslant \varepsilon$$

$$\max\nolimits_{t \geqslant 0} \{y_a(t; \alpha, \tau, \mathbf{q}^i)\} \leqslant (1 + 0.01\bar{S})\, y_1$$

$$\max\nolimits_{t \geqslant 0} \{|u_a(t; \alpha, \tau, \mathbf{q}^i)|\} \leqslant u_{\text{sat}}$$

$$t_s(\alpha, \tau, \mathbf{q}^i) \leqslant \eta$$

A practical solution to (48) can be found by using genetic algorithm tailored for constrained optimization. However, once a solution given by $(\alpha^*, \tau^*, \eta^*)$ has been obtained, it is necessary to validate it for robustness. Firstly, consider the robust stability. The polynomial $\xi(s; \alpha^*, \mathbf{q})$ has to be Hurwitz stable for all $\mathbf{q} \in \mathcal{Q}$. This analysis can be rigorously performed with the branch-and-bound algorithm of Balakrishnan *et al.* [21] or the interval procedure of Piazzi and Marro [22]. Secondly, verify that

$$\max_{t \geqslant 0,\ \mathbf{q} \in \mathcal{Q}} \{y_a(t; \alpha^*, \tau^*, \mathbf{q})\} \leqslant (1 + 0.01\bar{S})\, y_1 \tag{49}$$

$$\max_{t \geqslant 0,\ \mathbf{q} \in \mathcal{Q}} \{|u_a(t; \alpha^*, \tau^*, \mathbf{q})|\} \leqslant u_{\text{sat}} \tag{50}$$

Finally, compute

$$t_s^* := \max_{\mathbf{q} \in \mathcal{Q}} \{t_s(\alpha^*, \tau^*, \mathbf{q})\} \tag{51}$$

A naive method to solve the maximization problems in (49)–(51) is to perform a gridding of $\mathcal{Q}$ with extensive simulations. More efficient methods can be again found in stochastic global optimization (simulated annealing, genetic algorithms, etc. [23]) Suppose that robust stability is satisfied and that both inequalities (49) and (50) hold: it can happen that $t_s^* \geqslant \eta^*$. This means that the actual (suboptimal) worst-case settling-time is t_s^* as it results using the $C(s; \alpha^*)$ controller with reference input $r(t; \alpha^*, \tau^*)$. However, it is worth noting that expectedly t_s^* should not be much greater that τ^*. A serious drawback would be the failing of conditions (49) and (50) and/or of the robust stability requirement. In this case it would be necessary to deal with the more complex algorithms for solving (25).

Remark 3

From the above methodology, it appears how an optimization procedure has to be performed for each amplitude of the desired set-point transition. This requires that all the possible set-point values have to be known in advance for a specific plant.

Although this is quite common in practical situations, it has to be stressed that, in any case, parameter α and τ can be chosen in order to minimize the settling time assuring that saturation limits are not exceeded when the amplitude of the set-point transition is maximum. These values can then applied in a suboptimal way in the other cases. Another approach is to fix the value of α and then optimize the choice of τ only in the different situations.

7. SETTING-UP A COMPARATIVE DESIGN

With the aim of comparing the new method, exposed in the previous sections, with the classic unity-feedback control we set up the following design procedure which congruently acquires the same control specification given in Section 2. First, denote by $C(s; \mathbf{k})$ a chosen fixed-structure controller where $\mathbf{k} := [k_1, \ldots, k_d]^{\mathrm{T}}$ is a parameter design vector belonging to the given $\mathscr{K} := [k_1^-, k_1^+] \times \cdots \times [k_d^-, k_d^+] \subseteq \mathbb{R}^d$. In this context the considered design problem is: determine a (sub-)optimal $k^* \in \mathscr{K}$ such that

1. the closed-loop system is stable for all $\mathbf{q} \in \mathscr{Q}$;
2. $\lim_{t\to\infty} y(t) = y_1$ for all $\mathbf{q} \in \mathscr{Q}$ with $y(t)$ being the output response to the step reference $r(t) := y_1 1(t)$;
3. the overshoot in response to the step reference is bounded by a given $\bar{S}$ for all $\mathbf{q} \in \mathscr{Q}$.
4. the absolute value of the manipulative input $u(t)$ is bounded by a given u_{sat} for all $\mathbf{q} \in \mathscr{Q}$.
5. it is minimized the worst-case settling-time.

Note that the above five requirements are the same of those given in Section 2 apart from the fact that, here in classic design, the reference signal is known in advance and fixed to $r(t) := y_1 1(t)$. The steady-state condition of step 2 is secured by choosing a controller $C(s; \mathbf{k})$ which satisfies the following assumption.

Assumption 1

The unity-feedback system is of type 1 for all $\mathbf{q} \in \mathscr{Q}$ and all $\mathbf{k} \in \mathscr{K}$.

Let us denote by $\xi(s; \mathbf{k}, \mathbf{q}) := \sum_{i=0}^{n_c} \xi_i(\mathbf{k}, \mathbf{q})\, s^i$ the characteristic polynomial associated to the closed-loop system; Also denote by $H_i(\mathbf{k}, \mathbf{q})$ the Hurwitz determinant of $\xi(s; \mathbf{k}, \mathbf{q})$. The closed-loop structure of the controller has also to satisfy this assumption.

Assumption 2
The coefficient $\xi_{n_c}(\mathbf{k}, \mathbf{q})$ is always positive for all $\mathbf{q} \in \mathscr{Q}$ and all $\mathbf{k} \in \mathscr{K}$.

The stability robustness is taken into account through this result which is again derived from Lienard and Chipard's criterion [18].

Property 5
The closed-loop system is stable for all $\mathbf{q} \in \mathscr{Q}$ if and only if the following inequalities hold, for all $\mathbf{q} \in \mathscr{Q}$:

$$\xi_0(\mathbf{k}, \mathbf{q}) > 0; \quad \xi_1(\mathbf{k}, \mathbf{q}) > 0, \quad \xi_3(\mathbf{k}, \mathbf{q}) > 0, \ldots, \xi_v(\mathbf{k}, \mathbf{q}) > 0$$

$$H_{n_c-1}(\mathbf{k}, \mathbf{q}) > 0, \quad H_{n_c-3}(\mathbf{k}, \mathbf{q}), \ldots, H_w(\mathbf{k}, \mathbf{q}) > 0$$

with $v := n_c - 1$, $w := 3$ if n_c is even, while $v := n_c - 2$, $w := 2$ if n_c is odd.

Analogously, to the definitions given in Section 6, consider, corresponding to parameter vectors $\mathbf{k}$ and $\mathbf{q}$, the manipulative input $u_a(t; \mathbf{k}, \mathbf{q})$, the controlled output $y_a(t; \mathbf{k}, \mathbf{q})$, and the settling-time $t_s(\mathbf{k}, \mathbf{q})$. The posed design problem is then equivalent to the following optimization problem (with ε being a sufficiently small positive value):

$$\min_{\mathbf{k} \in \mathscr{K}} \max_{\mathbf{q} \in \mathscr{Q}} \{t_s(\mathbf{k}, \mathbf{q})\} \tag{52}$$

subject to, for all $\mathbf{q} \in \mathscr{Q}$:

$$\xi_0(\mathbf{k}, \mathbf{q}) \geqslant \varepsilon; \quad \xi_1(\mathbf{k}, \mathbf{q}) \geqslant \varepsilon, \quad \xi_3(\mathbf{k}, \mathbf{q}) \geqslant \varepsilon, \ldots, \xi_v(\mathbf{k}, \mathbf{q}) \geqslant \varepsilon$$

$$H_{n_c-1}(\mathbf{k}, \mathbf{q}) \geqslant \varepsilon, \quad H_{n_c-3}(\mathbf{k}, \mathbf{q}) \geqslant \varepsilon, \ldots, H_w(\mathbf{k}, \mathbf{q}) \geqslant \varepsilon$$

$$y_a(t; \mathbf{k}, \mathbf{q}) \leqslant (1 + 0.01\bar{S})\, y_1 \quad \forall t \geqslant 0$$

$$|u_a(t; \mathbf{k}, \mathbf{q})| \leqslant u_{\text{sat}} \quad \forall t \geqslant 0$$

In general, the above minimax problem (52) is more difficult than problem (25) because the dimension d of the parameter design space is typically greater than 2, i.e. the corresponding dimension of the search space in (25), and the semi-infinite inequalities related to the robust stability are more complex for problem (52) with respect to those for problem (25). Moreover, for problem (52), it can happen that no feasible $\mathbf{k}$ exists satisfying all the semi-infinite inequalities. This is due to the fact that we cannot claim, in this context, the validity of a result as Theorem 1. When this is the case, a new choice for the controller structure and/or a suitable enlargement of the 'box' domain $\mathscr{K}$ have to be considered. This drawback of the classic design over the new one is related to the use of the discontinuous reference signal $y_1 1(t)$ whereas in the new approach we use the arbitrarily smooth reference $r(t; \alpha, \tau)$.

For ease of computation, problem (52) can be relaxed by replacing $\mathscr{Q}$ with the finite set of its vertexes as it has been done in the previous section for problem (25). For brevity, details are omitted.

Remark 4
A possible way to improve the performance of the classic design is the inclusion of a prefilter for the reference step input. Besides the increased difficulty of the corresponding minimax problem (52), mainly due to the necessity of fixing the additional prefilter parameters, the foreseeable

worst-case performance is still inferior to that of the dynamic inversion-based controller. This point is highlighted is Section 8.

8. AN ILLUSTRATIVE EXAMPLE

As an illustrative example we consider the following uncertain plant ($h = 0$):

$$P(s; \mathbf{q}) = q_3 \frac{1 + q_1 s}{(s^2 + 2q_2 s + 1)(1.5s + 1)} \tag{53}$$

where $\mathbf{q}^0 = [0.5, 0.4, 3]$ and $\mathbf{q} = [q_1, q_2, q_3] \in \mathcal{Q} = [0.4, 0.6] \times [0.35, 0.45] \times [2.5, 3.5]$. We also have fixed $y_1 = 1$, $\bar{S} = 5$ per cent and $u_{\text{sat}} = 5$, so that satisfying condition (30) of Theorem 1. Optimizations has been performed using genetic algorithms [24].

8.1. Dynamic inversion based regulation

Following the new approach based on dynamic inversion, as explained in the previous sections, we set $a(s; \mathbf{q}^0) = (s^2 + 2q_2^0 s + 1)(1.5s + 1) = (s^2 + 0.8s + 1)(1.5s + 1)$ and $b(s; \mathbf{q}^0) = q_3^0(1 + q_1^0 s)$. For the stable user-chosen polynomial $d(s) = s + d_1$, we fix $d_1 = 20$ and it has to be noted that the choice of the value of d_1 is not a critical issue in the example, provided that this additional pole is sufficiently far in the left half-plane, as it will be shown in the next. Hence, we have

$$C(s; \alpha) = \alpha \frac{(s^2 + 0.8s + 1)(1.5s + 1)}{3s(1 + 0.5s)(s + 20)} \tag{54}$$

In order to have $r(t; \alpha, \tau) \in C^{(0)}$, a fifth-order polynomial ($p = 2$) has been chosen as output function (cf. Remark 2), so that we have

$$y(t; \tau) = \frac{6}{\tau^5} t^5 - \frac{15}{\tau^4} t^4 + \frac{10}{\tau^3} t^3 \quad \text{if } t \in [0, \tau]$$

and

$$y(t; \tau) = 1 \quad \text{if } t \in (\tau, \infty)$$

Consequently, the reference input, according to (24) is

$$r(t; \alpha, \tau) = \begin{cases} \dfrac{6}{\tau^5} t^5 - \dfrac{15}{\tau^4} t^4 + \dfrac{10}{\tau^3} t^3 & \\ \quad + \dfrac{1}{\alpha} \left[20 \left(\dfrac{30}{\tau^5} t^4 - \dfrac{60}{\tau^4} t^3 + \dfrac{30}{\tau^3} t^2 \right) + \dfrac{120}{\tau^5} t^3 - \dfrac{180}{\tau^4} t^2 + \dfrac{60}{\tau^3} t \right] & \text{if } t \in [0, \tau] \\ 1 & \text{if } t \in (\tau, \infty) \end{cases}$$

The values found by the genetic algorithms are $\alpha^* = 589.4$ and $\tau^* = 1.26$ which results in an optimal worst-case settling-time t_s^* equal to 1.12 s. The resulting reference input $r(t; \alpha^*, \tau^*)$ is plotted

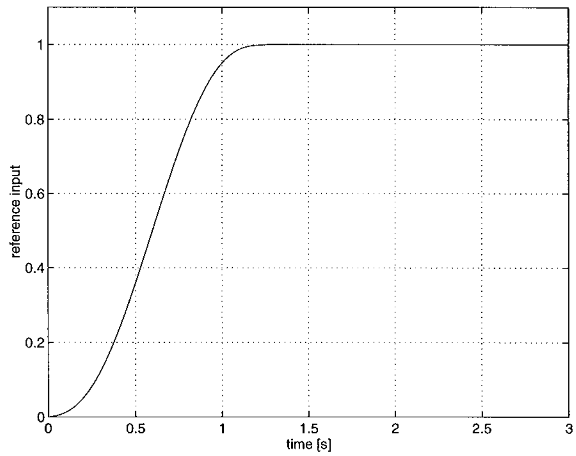

Figure 2. The resulting reference input determined by inverting the system.

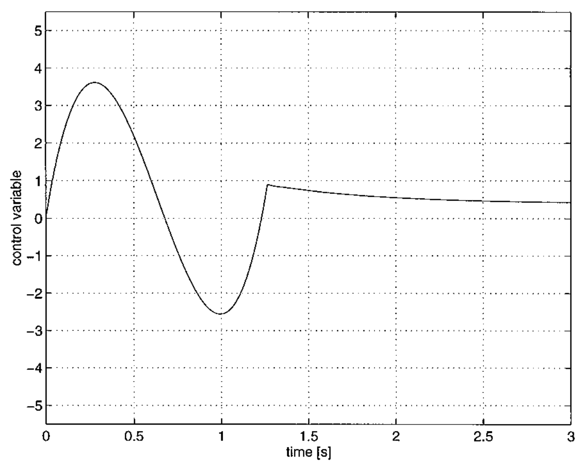

Figure 3. Control variable of the optimal worst-case settling time with the dynamic inversion-based controller.

in Figure 2. Figures 3 and 4 shows the control variable and the output signals corresponding to the worst-case settling time, where $q_1 = 0.6$, $q_2 = 0.45$ and $q_3 = 2.5$. Figures 5 and 6 report the control variable and the system output respectively, for the case in which the control variable limit is achieved (this happens when $q_1 = 0.4$, $q_2 = 0.45$ and $q_3 = 2.5$). It has to be noted that the limit of the overshoot is never achieved with the optimal setting of the parameters α and τ. In other words, the overshoot constraint is not active in the optimal solution. Finally, we applied the optimizations to the dynamic inversion-based regulation scheme as above but with $d_1 = 30$. The resulting optimal worst-case settling time was again equal to 1.12 s.

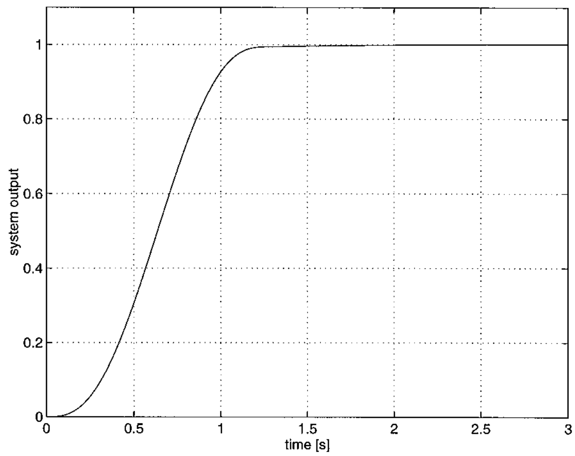

Figure 4. System output of the optimal worst-case settling time with the dynamic inversion-based controller.

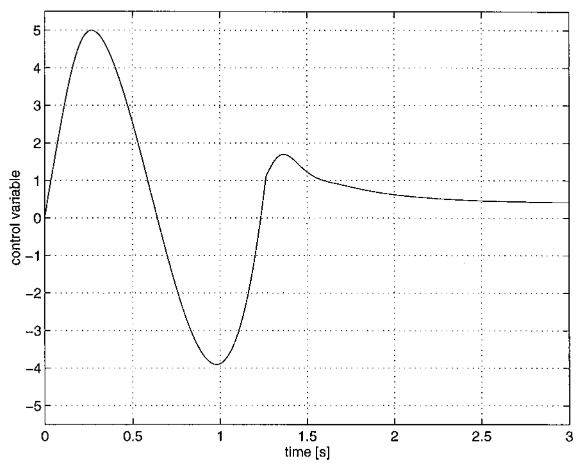

Figure 5. Control variable with the dynamic inversion based controller when the saturation limit is achieved.

8.2. Traditional controller design

To make a comparison, we choose the following fixed-structure controller, having the same order of the one determined in the dynamic inversion-based approach:

$$C(s; \mathbf{k}) = k_1 \frac{(1 + 2k_2 \ s/k_3 + s^2/k_3^2)(1 + k_4 s)}{s(1 + as)(1 + bs)} \tag{55}$$

where a has been fixed to 0.05 s and b to 0.04 s. We also consider $\mathscr{K} = [0.01, 5] \times [0.1, 3] \times [0.1, 5] \times [0.1, 5]$. The found optimal design parameters are $k_1^* = 0.12$, $k_2^* = 1.67$, $k_3^* = 4.12$ and

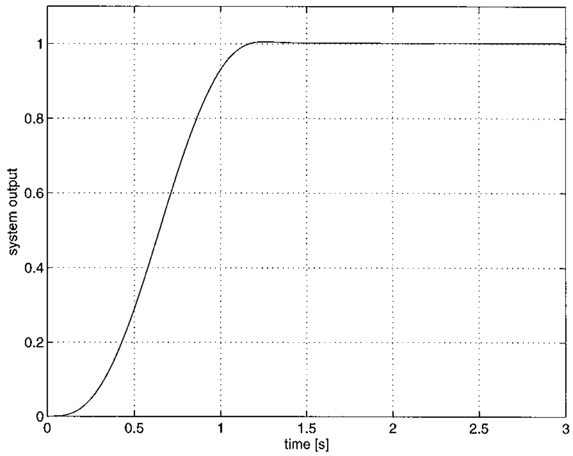

Figure 6. System output with the dynamic inversion-based controller when the saturation limit is achieved.

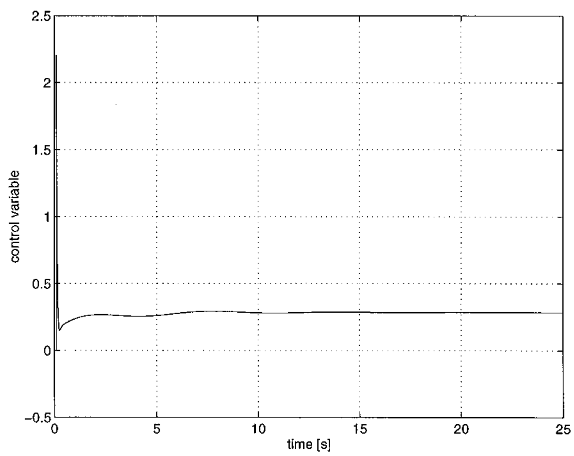

Figure 7. Control variable with the traditional controller for the optimal worst-case settling time.

$k_4^* = 0.59$. It results that the optimal worst-case settling-time t_s^* is equal to 11.22 s. This settling time is actually obtained when $q_1 = 0.4$, $q_2 = 0.35$ and $q_3 = 3.5$. The corresponding plots of the control variable and output signals are shown in Figures 7 and 8, respectively. Note that in this case the unit step input starts at 0.1 s and that the axis scaling of the control variable is different from the case of the dynamic inversion approach.

It has to be noted that even filtering the set-point in the traditional controller design does not yield to the same performances as in the dynamic-inversion-based case. To prove this, we considered a control scheme (see Figure 9) in which controller (55) has been employed and, in addition, the step reference input has been filtered by means of a second order system

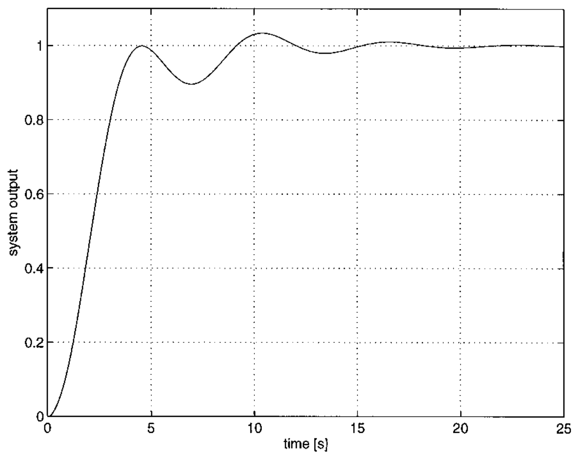

Figure 8. System output with the traditional controller for the optimal worst-case settling time.

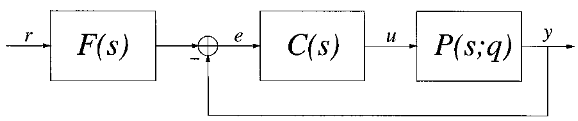

Figure 9. Traditional controller design with filtered reference input.

with transfer function

$$F(s; \mathbf{f}) = \frac{1}{(f_1 s + 1)(f_2 s + 1)} \tag{56}$$

The six free parameters k_i, $i = 1, \dots, 4$ and f_j, $j = 1, 2$ have been again found again by means of genetic algorithms. The resulting values are $k_1^* = 10.81$, $k_2^* = 1.94$, $k_3^* = 3.98$, $k_4^* = 0.87$, $f_1^* = 0.37$, $f_2^* = 0.38$. The optimal worst-case settling time is 2.54 s, which occurs when $q_1 = 0.4$, $q_2 = 0.35$ and $q_3 = 2.5$. Plots of the filtered reference input, of the control variable and of the system output in the worst-case are shown in Figures 10, 11 and 12, respectively.

8.3. Discussion

In both cases 8.1 and 8.2 the optimization step has been applied to the 'relaxed' problems. The necessary computation of the optimal worst-case t_s^* and the verification of robust performances associated to overshoot limitation and to the avoidance of the control saturation have been again done using genetic algorithms for both cases (cf. (49)–(51)). Comparing the optimal worst-case settling times, a great improvement obtained with the new approach emerges. It appears that the synthesized dynamic inversion-based controller (54) is a high-gain controller compared with the traditional one (55) without the prefilter. Indeed, the velocity constants in the open-loop are respectively, for the nominal plant, $K_v = 29.5$ for (54) and for the classic design (55), $K_v = 0.36$ (absence of the prefilter) and $K_v = 32.43$ (presence of the prefilter). Note that, although the

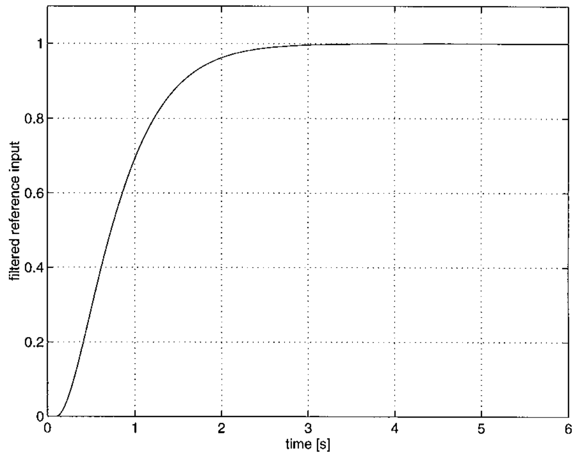

Figure 10. Filtered reference input with the traditional controller for the optimal worst-case settling time.

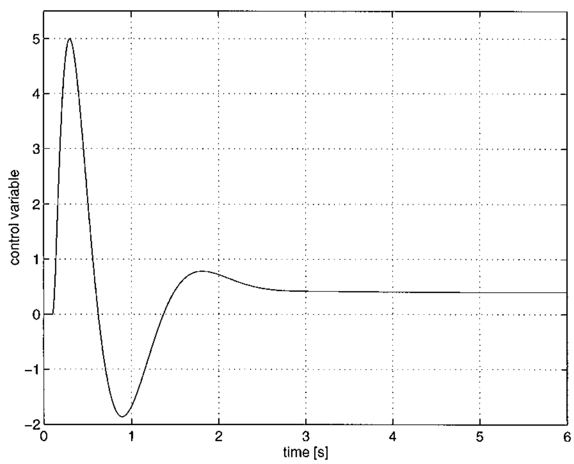

Figure 11. Control variable with the traditional controller with filtered reference input for the optimal worst-case settling time.

velocity constants of the traditional design with prefilter and of the new design are similar, the worst-case settling time of the dynamic inversion-based controller is still more than two times better over the improved traditional controller.

9. CONCLUSIONS

In this paper, a novel approach for the robust set-point regulation of a scalar system has been presented. The control system design methodology involves the determination of both the

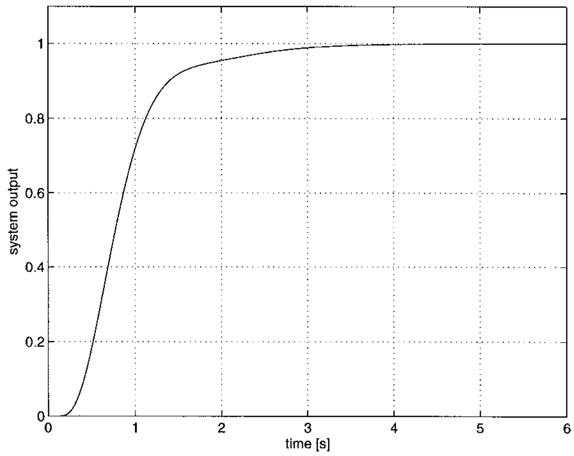

Figure 12. System output with the traditional controller with filtered reference input for the optimal worst-case settling time.

feedback controller and the reference input. Roughly stating, it consists of substituting the traditional step at the input with a reference function calculated through a dynamic system inversion after that a polynomial function has been assigned to the output. The design of the feedback controller is greatly simplified and it has been shown in an example that the new approach performs much better, in terms of the worst-case settling time, compared with a traditional design approach.

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