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# End-Point Control of a Flexible-Link via Optimal Dynamic Inversion

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*Abstract*—**In this paper we propose a new method for the end-point control of a single flexible link. As a distinguished feature, the methodology relies on an exact stable dynamic inversion, analytically performed on a family of output functions. In this context, the choice of "transition polynomials" as output functions permits to optimize the motion time subject to limits on the velocity and acceleration of the motor. Furthermore, the technique is robust to unmodelled dynamics. Simulation results show the effectiveness of the approach.**

## I. Introduction

Flexible manipulators have attracted much attention in the last twenty years because of the many advantages they exhibit with respect to the conventional rigid ones. In fact,they need less material, smaller actuators, less energy and at the same time they are capable to assure faster motions and a higher ratio of payload to arm weight. However, the control design task is much more difficult because, despite the fact that good results can be achieved with a linearized model (see for example [1]), many flexible modes are required to accurately model the system. In addition, the fact that the positioning sensor at the manipulator's tip is non colocated with the actuator at the hub renders the system nonminimum-phase. Hence, a direct application of a standard inverse control strategy leads to an internal instability of the system and it is therefore not possible in practical cases.

To achieve a stable inversion, many solutions have been proposed. For example, the concept of integral manifold has been exploited in [2], whilst in [3], [1] the principle of transmission zero assignment is adopted in an output feedback control strategy. The zero dynamics can also been stabilized by means of the output redefinition approach [4], [5], [6] or by considering a series expansion of the inverse of the nonminimum-phase zeros [7]. Finally, iterative learning control [8], [10] and neural networks [9] have been also successfully applied.

In this paper we present a methodology to analitically perform an exact stable inversion of the linearized model of the system. This allows to minimize the time interval of a point-to-point motion, tacking into account the actuator's limits. The optimal dynamic inversion relies on the concept of "transition polynomial" [11], which has been already exploited for the robust control of minimum-phase mechanical systems, namely, for the residual vibration reduction of a servopositioning system endowed with an elastic transmission [12]. Roughly stating, the technique consists of defining a priori, as a family of polynomial functions which depend on the motion time, the desired motion law of the robot's tip and then determining, by means of an exact stable inversion procedure, the corresponding motion law of the motor at the hub. The motion time is subsequently minimized subject to actuator's constraints. In this context, the control of the end-effector position is actually performed in open-loop, whilst a closed-loop controller is applied to the motor in order to track the calculated motion law. This choice has been done because, although with the proposed dynamic inversion methodology it is possible to compute the motor torque to be applied to the motor and to implement a control scheme with a feedback of the tip's position, in this way there is no need to measure the tip's position itself. Moreover, the modelling task is simplified, as it is related only to the flexible link and therefore problems such as the motor's friction can be handled at a control level, by using specific methodologies [1]. Note that an exact dynamic inversion approach has been also studied in [13], but in that case the robustness and optimization issues have not been addressed.

The paper is organized as follows. In Section II the transfer function of a single flexible link is determined and some notation is introduced. In Section III the problem is formulated and the methodology to solve it is presented. Simulation results are given in Section IV and conclusions are drawn in Section V.

## II. Transfer Function of a Single Flexible-Link

The single flexible link we consider is assumed to be a uniform beam, of length $h$ and mass density $\gamma$. One end of the link is connected to a motor and the other one is free. It is also assumed that the height of the link is much greater than its width, that is, the link moves and vibrates in the horizontal plane and the effects of the shear deformation are negligible. Considering a frame X-Y, defined such as the X-axis always passes through the center of gravity of the link, the position of any point along the link can be written as:

$$y(t) = x\theta(t) + w(x,t) \tag{1}$$

where $\theta(t)$ is the rotation of the hub and $w(x,t)$ is the elastic deflection (see Figure 1). Using the assumed-modes approach [14], [15], the elastic deflection can be expressed by:

$$w(x,t) = \sum_{i=1}^{n} q_i(t)\phi_i(x) \tag{2}$$

where $q_i(t)$ is the generalized coordinate of the $i$th mode and $\phi_i(x)$ is the normalized, clamped-free eigenfunction of the $i$th mode:

$$\phi_i(x) = c_i\left[\sin k_i x - \sinh k_i x - \frac{\sin k_i h + \sinh k_i h}{\cos k_i h + \cosh k_i h}(\cos k_i h - \cosh k_i h)\right] \tag{3}$$

where the $k_i$ are the solutions to:

$$\cos(k_i h)\cosh(k_i h) = -1$$

and the $c_i$ are normalization constants to be chosen so that

$$\int_0^h \phi_i(x)^2 dx = 1.$$

Starting from the Euler-Lagrange equations and following the derivations reported in [16], [17], [3], [1], we can write the equations of the non linear model of the system in which it is assumed that the hub position control is ideal, i.e. the actual hub position is equal to the desired one. The verification of this assumption is facilitated in practical cases by the presence of a speed reducer (as the inertia torque exerted on the hub by the flexible modes, i.e. the rigid body modes of the system and the flexible ones are decoupled) and by an appropriate selection of the controller's gain which makes the rigid body modes to decay much more rapidly than the flexible ones [3], [1]. Thus, considering the hub position $\theta$ as input of the system and the tip's position $y$ as output, it results:

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{K_1}\mathbf{q} = \dot{\theta}^2\mathbf{K_2}\mathbf{q} - \mathbf{m}\ddot{\theta}, \tag{4}$$

where:

$$\mathbf{M} = \begin{bmatrix} \gamma + M_p\phi_1^2(h) & M_p\phi_1(h)\phi_2(h) & \cdots & M_p\phi_1(h)\phi_n(h) \\ M_p\phi_2(h)\phi_1(h) & \gamma + M_p\phi_2^2(h) & \cdots & M_p\phi_2(h)\phi_n(h) \\ \vdots & \vdots & \vdots & \vdots \\ M_p\phi_n(h)\phi_1(h) & M_p\phi_n(h)\phi_2(h) & \cdots & \gamma + M_p\phi_n^2(h) \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} c_1 & 0 & \cdots & 0 \\ 0 & c_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & c_n \end{bmatrix}$$

$$\mathbf{K_1} = \gamma \begin{bmatrix} \omega_1^2 & 0 & \cdots & 0 \\ 0 & \omega_2^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \omega_n^2 \end{bmatrix}$$

$$\mathbf{K_2} = \begin{bmatrix} \gamma + M_p\phi_1^2(h) & M_p\phi_1(h)\phi_2(h) & \cdots & M_p\phi_1(h)\phi_n(h) \\ M_p\phi_2(h)\phi_1(h) & \gamma + M_p\phi_2^2(h) & \cdots & M_p\phi_2(h)\phi_n(h) \\ \vdots & \vdots & \vdots & \vdots \\ M_p\phi_n(h)\phi_1(h) & M_p\phi_n(h)\phi_2(h) & \cdots & \gamma + M_p\phi_n^2(h) \end{bmatrix}$$

$$\mathbf{m}^T = \left[\gamma\int_0^h \phi_1 x dx + M_p h\phi_1(h) \quad \gamma\int_0^h \phi_2 x dx + M_p h\phi_2(h) \cdots \cdots \gamma\int_0^h \phi_n x dx + M_p h\phi_n(h)\right]$$

In the previous expressions, $M_p$ is the mass of the load, $\omega_i$ is the resonance frequency of the $i$th clamped-free eigenfunction and the terms $c_i\dot{q}_i$ $(i = 1, \ldots, n)$ describe the structural damping of the arm.

By linearizing equation (4) around the operating point $\dot{\theta}_0 = \ddot{\theta}_0 = 0$ and $\mathbf{q_0} = \dot{\mathbf{q}}_0 = \ddot{\mathbf{q}}_0 = 0$ we obtain [3], [1]:

$$\begin{array}{rcl} \dot{\mathbf{v}}(t) & = & \mathbf{A}\mathbf{v}(t) + \mathbf{b}\delta\theta(t) \\ \delta y(t) & = & \mathbf{c}\mathbf{v}(t) + d\delta\theta(t) \end{array} \tag{5}$$

where

$$\mathbf{v} = \begin{bmatrix} \delta\mathbf{q} + \mathbf{M}^{-1}\mathbf{m}\delta\theta \\ \delta\dot{\mathbf{q}} + \mathbf{M}^{-1}\mathbf{m}\delta\dot{\theta} + \mathbf{M}^{-1}\mathbf{C}\delta\mathbf{q} \end{bmatrix}$$

is the state vector where $\delta\mathbf{q}$, $\delta\dot{\mathbf{q}}$, $\delta\theta$ and $\delta\dot{\theta}$ denote small perturbations of $\mathbf{q}$, $\dot{\mathbf{q}}$, $\theta$ and $\dot{\theta}$ respectively around the operating point and

$$\mathbf{A} = \begin{bmatrix} -\mathbf{M}^{-1}\mathbf{C} & \mathbf{I_n} \\ -\mathbf{M}^{-1}\mathbf{K_1} & \mathbf{0_n} \end{bmatrix}$$

$$\mathbf{b} = \begin{bmatrix} \mathbf{M}^{-1}\mathbf{C}\mathbf{M}^{-1}\mathbf{m} \\ \mathbf{M}^{-1}\mathbf{K_1}\mathbf{M}^{-1}\mathbf{m} \end{bmatrix}$$

$$\mathbf{c} = [\Phi(h) \;\; \mathbf{0_{1\times n}}]$$

$$d = h - \Phi(h)\mathbf{M}^{-1}\mathbf{m}$$

where

$$\Phi(h) = [\phi_1(h) \; \phi_2(h) \cdots \phi_n(h)].$$

For the purpose of applying the design methodology described in Section III it is convenient to express the system (5) in the transfer function form

$$G(s) = \mathbf{c}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{b} + d = \frac{b(s)}{a(s)}. \tag{6}$$

Note that $G(s)$ is biproper.

## III. Control via Optimal Dynamic Inversion

### A. Exact stable dynamic inversion

The objective of the dynamic inversion based procedure is to determine a bounded input function (i.e. a motion law of the motor at the hub) which determines a desired output function (i.e. a motion law of the end-effector from a position $y_0$ to $y_1$). Without loss of generality, in the following we will assume $y_0 = 0$. The set of all cause/effect function pairs $(\theta(\cdot), y(\cdot))$ associated to $G(s)$ be denoted by $\mathcal{B}$. The desired end-effector motion law is defined in the

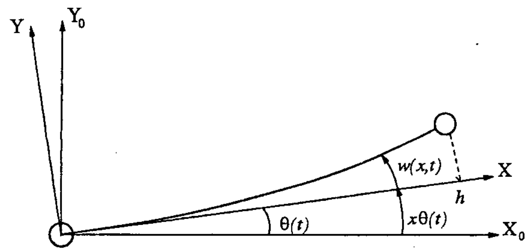


Fig. 1. Sketch of the flexible link.

time interval $[0,\tau]$ by means of the "transition" polynomials introduced in [11]:

$$y(t;\tau)=y_1\frac{(2l+1)!}{l!\tau^{2l+1}}\sum_{i=0}^{l}\frac{(-1)^{l-i}}{i!(l-i)!(2l-i+1)}\tau^{i}t^{2l-i+1}. \tag{7}$$

Outside the interval $[0,\tau]$, $y(t;\tau)$ is simply defined as $y(t;\tau)=0$ if $t\le 0$ and $y(t;\tau)=y_1$ if $t\ge\tau$. In (7) the integer $l$ can be arbitrarily chosen in order to assure $y(t;\tau)\in C^{(l)}$ over $\mathbb{R}$, i.e. $y(t;\tau)$ has continuous derivatives up to the $l$th order. An alternative expression of (7) is given by the integral representation [11]

$$y(t;\tau)=y_1\frac{(2l+1)!}{(l!)^2\tau^{2l+1}}\int_0^t v^l(\tau-v)^l dv,\quad t\in[0,\tau]. \tag{8}$$

that makes clear that $y(t;\tau)$ is monotonically increasing since $\dot{y}(t;\tau)$ is positive over $(0,\tau)$. As a consequence, the planned motion of the end-effector is, by construction, free of oscillatory modes.

Note that, to select an appropriate value of $l$, the following proposition, whose proof is a straightforward consequence of the one in [18], has to be exploited:

*Proposition 1:* For the system described by the transfer function (6), consider a pair $(\theta(\cdot),y(\cdot))\in\mathcal{B}$. Then $\theta(\cdot)\in C^{(l)}$ if and only if $y(\cdot)\in C^{(l)}$ with $l$ being a non negative integer.

Hence, a suitable choice of $l$ allows to determine a continuous velocity reference function $\dot{\theta}(t)$ and in case also a continuous acceleration reference function $\ddot{\theta}(t)$.

Now, in order to perform the stable inversion, we rewrite the numerator of the transfer function (6) as follows:

$$b(s)=b_-(s)b_+(s)$$

where $b_-(s)$ and $b_+(s)$ denote the polynomials associated to the zeros with negative real part and positive real part respectively (appropriately, the presence of purely imaginary zeros can excluded). Now, consider the inverse system of (6) whose transfer function can be written as:

$$G(s)^{-1}=\gamma_0+H_0(s)$$

where $\gamma_0$ is a real constant and $H_0(s)$, a strictly proper rational function, represents the zero dynamics. This can be uniquely decomposed according to

$$H_0(s)=H_0^-(s)+H_0^+(s)=\frac{c(s)}{b_-(s)}+\frac{d(s)}{b_+(s)}$$

where $c(s)$ and $d(s)$ are suitable polynomials. Being $\mathcal{L}$ the Laplace transform operator, define:

$$\eta_0^-(t):=\mathcal{L}^{-1}[H_0^-(s)]$$
$$\eta_0^+(t):=\mathcal{L}^{-1}[H_0^+(s)]$$
$$Y(s;\tau):=\mathcal{L}[y(t;\tau)]$$

The unstable reference function $\theta_u(t;\tau)$ that causes the desired output function $y(t;\tau)$ can be simply determined as:

$$\begin{aligned}\theta_u(t;\tau) &= \mathcal{L}^{-1}[G(s)^{-1}Y(s;\tau)]\\ &= \gamma_0 y(t;\tau)+\textstyle\int_0^t\eta_0^-(t-v)y(v;\tau)dv+\\ &\quad \textstyle\int_0^t\eta_0^+(t-v)y(v;\tau)dv\\ &\quad t\in(-\infty,+\infty).\end{aligned} \tag{9}$$

Thus, we have that $(\theta_u(t;\tau),y(t;\tau))\in\mathcal{B}$ and note that $\theta_u(t;\tau)=0$ if $t\in(-\infty,0)$ and $\theta_u(t;\tau)$ is unbounded over $[0,+\infty)$ due to the unstable zero dynamics (associated to $H_0^+(s)$).

The unstable modes associated with $b_+(s)$ be denoted by $m_i(t)$, $i=1,\dots,w$. Then, the following lemma results:

*Lemma 1:* There exists real constants $k_i\in\mathbb{R}$, $i=0,\dots,w$ that depend on positive time parameter $\tau$ such that, for $t>\tau$

$$\int_0^t\eta_0^+(t-v)y(v;\tau)dv=k_0(\tau)+\sum_{i=1}^{w}k_i(\tau)m_i(t).$$

*Proof.* Considering that $t>\tau$ we can rewrite the integral of the above Lemma as follows:

$$\begin{aligned}\textstyle\int_0^t\eta_0^+(t-v)y(v;\tau)dv &= \textstyle\int_0^\tau\eta_0^+(t-v)y(v;\tau)dv+\\ &\quad\textstyle\int_\tau^t\eta_0^+(t-v)y_1dv.\end{aligned} \tag{10}$$

As it is known $\eta_0^+(t)$, the impulse response of the unstable zero dynamics can be expressed as a linear combination of the modes $m_i(t)$:

$$\eta_0^+(t)=\alpha_1 m_1(t)+\cdots+\alpha_w m_w(t) \tag{11}$$

where $\alpha_i\in\mathbb{R}$, $i=1,\dots,w$ are appropriate coefficients. Taking into account the analytic expression of the transition polynomial $y(t;\tau)$ it then follows that the integral $\int_0^\tau\eta_0^+(t-v)y(v;\tau)dv$ is a linear combination of the modes $m_i(t)$ and its coefficients depend on $\tau$. On the other hand, examining the integral $\int_\tau^t\eta_0^+(t-v)y_1dv$ we analogously deduce that it can be expressed as a linear combination of the modes $m_i(t)$ plus a constant addend. Therefore, by virtue of (10) the statement of Lemma 1 follows. □

At this point, taking into account Lemma 1 we can define the following function:

$$\theta_c(t;\tau):=-\sum_{i=1}^{w}k_i(\tau)m_i(t)\quad t\in(-\infty,+\infty). \tag{12}$$

The following lemma results:

*Lemma 2:* Being $\theta_c(t;\tau)$ the function defined in (12), we have

$$(\theta_c(t;\tau),0)\in\mathcal{B}\ \ \forall\tau\in\mathbb{R}^+.$$

*Proof.* By examination of the differential equation associated to the system described by the transfer function $G(s) = (b_-(s)b_+(s))/a(s)$ it follows that the pair $(\theta_c(t;\tau), 0)$ satisfies this equation over $(-\infty, +\infty)$. □

Finally, we can define the following function, which perform the exact stable inversion:

$$\theta(t;\tau) = \theta_u(t;\tau) + \theta_c(t;\tau) \quad t \in (-\infty, +\infty). \tag{13}$$

The following proposition can therefore be stated.

*Proposition 2:* The function function $\theta(t;\tau)$ defined in (13) is bounded over $(-\infty, +\infty)$ and $(\theta(t;\tau), y(t;\tau)) \in \mathcal{B}$.

*Proof.* Taking into account Lemma 2, evidently $(\theta(t;\tau), y(t;\tau)) \in \mathcal{B}$ by virtue of linear superposition. On the other hand, $\theta(t;\tau)$ is, by construction, bounded because of the exact cancellation of all the unstable modes appearing in $\theta_u(t;\tau)$ (see Lemma 1 and definition (12)). □

Summarizing, the determined function $\theta(t;\tau)$ exactly solves the stable inversion problem for a family of output functions, which depend on the free transition time $\tau$.

### B. Optimization

By virtue of the stable exact dynamic inversion procedure exposed in the previous subsection, we can now pose the following optimization problem, which aims at minimizing the motion time subject to actuators constraints:

$$\min_{\tau \in \mathbb{R}^+} \tau \tag{14}$$

subject to:

$$\theta_p \leq \theta(t;\tau) \leq \theta_n \quad \forall t \in (-\infty, +\infty) \tag{15}$$

$$|\dot{\theta}(t;\tau)| \leq \dot{\theta}_{MAX} \quad \forall t \in (-\infty, +\infty) \tag{16}$$

$$|\ddot{\theta}(t;\tau)| \leq \ddot{\theta}_{MAX} \quad \forall t \in (-\infty, +\infty) \tag{17}$$

where $\dot{\theta}_{MAX}$ and $\ddot{\theta}_{MAX}$ have to be selected based on the actuators limits and $\theta_p$ and $\theta_n$ are undershooting and overshooting limits. In particular, the constraint (16) is evidently related a velocity limit of the actuator whilst (17) is related to a torque limit.

*Proposition 3:* The optimization problem (14) admits a solution if $\dot{\theta}_{MAX} > 0$, $\ddot{\theta}_{MAX} > 0$, and $\theta_p < 0$ and $\theta_n > \theta_f$ where $\theta_f := y_1/G(0)$.

*Proof.* For brevity the proof is omitted. It is an extension of a proof appearing in [19].

An effective way to find the optimal solution $\tau^*$ of the above optimization problem is to use a simple bisection algorithm, as follows:

1. Set $\tau_{min} = 0$.
2. Determine an initial value for $\tau_{max}$ such as constraints (15)-(16)-(17) are satisfied.
3. Set $\tau = (\tau_{min} + \tau_{max})/2$.
4. If constraints (15)-(16)-(17) are satisfied then set $\tau_{max} = \tau$ else set $\tau_{min} = \tau$.
5. If $(\tau_{max} - \tau_{min}) > \varepsilon$ then goto 3.
6. Set $\tau^* = \tau_{max}$.
7. End.

*Remark 1.* At step 4, checking if the constraints are satisified requires the computation of the global maximum and minimum of $\theta(t;\tau)$ and its first and second time derivatives over the unbounded domain $(-\infty, +\infty)$. This can be successfully performed by means of a suitable interval algorithm as reported in [20, Chapter 4].

The initial value of $\tau_{max}$ can be easily found starting from a reasonable value and if it doesn't satisfy the constraints of point 2, multiplying it repeatedly by a constant $\lambda > 1$ until the condition is true. The other precision parameter $\varepsilon > 0$ determines the terminal condition of the algorithm at point 5.

Actually, from a practical point of view, in order to use the synthesized function (13) with the optimal $\tau^*$ it is necessary to truncate it, resulting therefore in an approximate generation of the desired output $y(t;\tau)$. This can be done with arbitrarily precision given any couple of small parameters $\varepsilon_0 > 0$ and $\varepsilon_1 > 0$. Indeed, compute

$$t_0 := \max\{t' \in \mathbb{R} : |\theta(t;\tau^*)| \leq \varepsilon_0 \quad \forall t \in (-\infty, t']\}$$

and define

$$t_s := \min\{0, t_0\}.$$

Similarly, compute

$$t_f := \min\{t' \in \mathbb{R} : |\theta(t;\tau^*) - \theta_f| \leq \varepsilon_1 \quad \forall t \in [t', \infty)\}.$$

Hence, the approximate reference signal to be used is

$$\theta_a(t;\tau^*) := \begin{cases} 0 & \text{for} \quad t < t_s \\ \theta(t;\tau^*) & \text{for} \quad t_s \leq t \leq t_f \\ \theta_f & \text{for} \quad t > t_f. \end{cases}$$

Note that $t_s$ depends on $\tau$ and it might occur that $t_s < 0$, resulting in the so-called "preaction control" [21], [22].

## IV. Simulation Results

To evaluate the proposed methodology, we simulated the flexible link whose model has been presented in [3], [1]. Four flexible modes have been retained for the purpose of simulation, but, actually, only the first has been considered to obtain the linearized model that has been inverted to determine the motion law of the hub. The zeros and the poles of the transfer function $G(s)$ when four flexible modes have been considered are reported in Table I, whilst those of the model adopted for the exact dynamic inversion procedure are reported in Table II. Thus, the proposed methodology has been applied to a simple second order (nonminimum-phase) system with real zeros. A motion of $y_1 = 0.1$m has been imposed to the end-effector, according to a fifth-order polynomial function, in order to assure the continuity of the position, velocity and acceleration law of the hub, i.e.

$$y(t;\tau) = 0.1\left(\frac{6}{\tau^5}t^5 - \frac{15}{\tau^4}t^4 + \frac{10}{\tau^3}t^3\right) \quad t \in [0, \tau].$$

The optimization algorithm has been applied after having fixed $\theta_p = -1 \cdot 10^{-3}$rad, $\theta_n = 0.084$rad (note that $\theta_f = 0.083$rad), $\dot{\theta}_{MAX} = 1$rad·s$^{-1}$ and $\ddot{\theta}_{MAX} = 2$rad·s$^{-2}$.

| | | | | | |
|---|---|---|---|---|---|
| Zeros | $-1.87 \pm 77.0i$ | $-1.12 \pm 47.9i$ | $14.3 \pm 4.0i$ | $-20.1$ | $-8.0$ |
| Poles | $-2.07 \pm 102.0i$ | $-0.83 \pm 52.0i$ | $-1.66 \pm 18.9i$ | | $0.17 \pm 3.0i$ |

TABLE I
POLES AND ZEROS OF THE LINEARIZED MODEL WITH FOUR FLEXIBLE MODES.

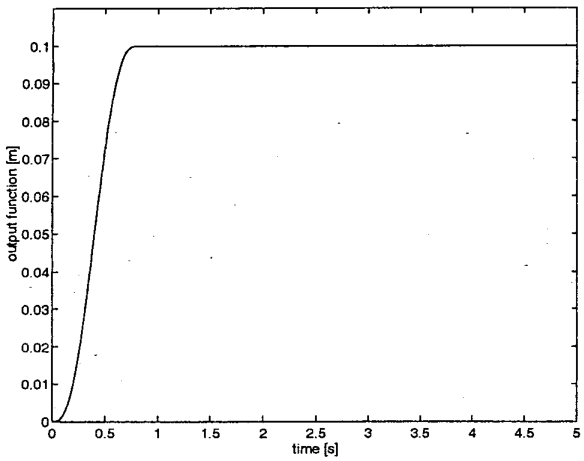


Fig. 2. Desired tip's motion law.

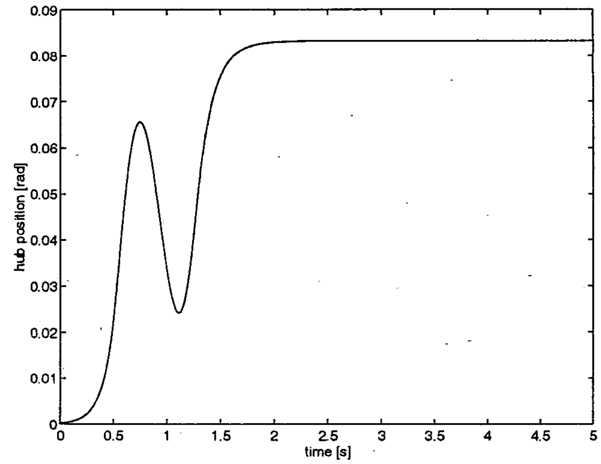


Fig. 3. Hub position reference command input.

| | | |
|---|---|---|
| Zeros | 9.31 | $-6.93$ |
| Poles | $-1.16 \pm 2.99i$ | |

TABLE II
POLES AND ZEROS OF THE LINEARIZED MODEL WITH ONE FLEXIBLE MODE.

The resulting optimal motion time is $\tau^* = 0.8$s and the corresponding desired tip's motion law is shown in Figure 2. Finally, we selected $\varepsilon_0 = \varepsilon_1 = 2 \cdot 10^{-4}$, that yields to a preaction time of 0.5s (i.e. $t_s = -0.5$s) and to an overall motion time of the hub of 2.1s (i.e. $t_f = 1.6$s). The hub position reference command is plotted in Figure 3, whilst in Figures 4 and 5 are reported the hub velocity and acceleration reference inputs respectively. It appears that the significant constraint is the one on the acceleration signal. The actual end-effector motion that results from the application of the calculated reference signal of the hub to the system is shown in Figure 6. It turns out that, despite only one flexible mode has been considered in the modeling phase, the residual oscillation is somewhat limited.

## V. Conclusions

In this paper, a new method, based on an exact stable inversion procedure, for the end-effector control of a single flexible link has been presented. Basically, it relies on an open-loop approach, so that the measurement of the tip's position is not required. In spite of this, the use of smooth functions assures an inherent robustness to the system, namely, the technique provides good performances even if only one flexible mode is considered in the modeling phase, which can be therefore significantly simplified. Furthermore, the motion time can be minimized taking into account actuator's constraints by means of a simple optimization algorithm.


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## References


[1] H. Geniele, R. V. Patel, K. Khorasani, "End-point control of a flexible-link manipulator: theory and experiments", *IEEE Transactions on Control Systems Technology*, Vol. 5, No. 6, pp. 556–570, 1997.

[2] K. Hashtrudi Zaad, K. Khorasani, "Control of non-minimum phase singularly perturbed systems with application to flexible-link manipulators", *International Journal of Control*, Vol. 63, No. 4, pp. 679–701, 1996.

[3] H. Geniele, R. V. Patel, K. Khorasani, "Control of a flexible link manipulator", *Proc. IEEE International Conference on Robotics and Automation*, pp. 1217–1222, 1995.

[4] M. Moallem, R. V. Patel, K. Khorasani, "An inverse dynamics control strategy for tip position tracking of flexible multi-link manipulators", *Journal of Robotic Systems*, Vol. 14, No. 9, pp. 649–658, 1997.

[5] H. A. Talebi, K. Khorasani, R. V. Patel, "Experimental results on tracking control of a flexible-link manipulator: a new output redefinition approach", *Proc. IEEE International Conference on Robotics and Automation*, pp. 1090–1095, 1999.

[6] H. Yang, H. Krishnan, M. H. Ang, "Tip-trajectory tracking control of single-link flexible robots via output redefinition", *Proc.*

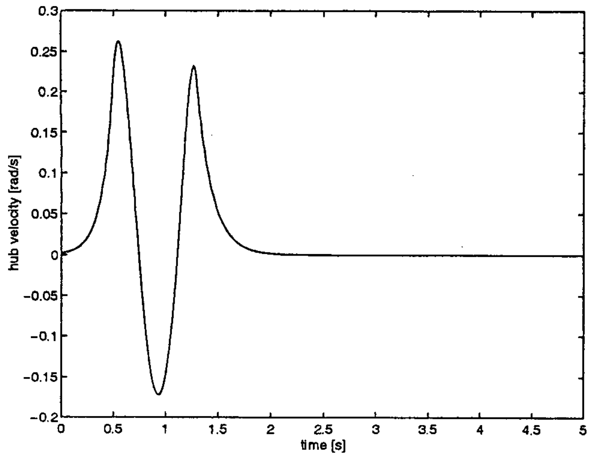


Fig. 4. Hub velocity reference command input.

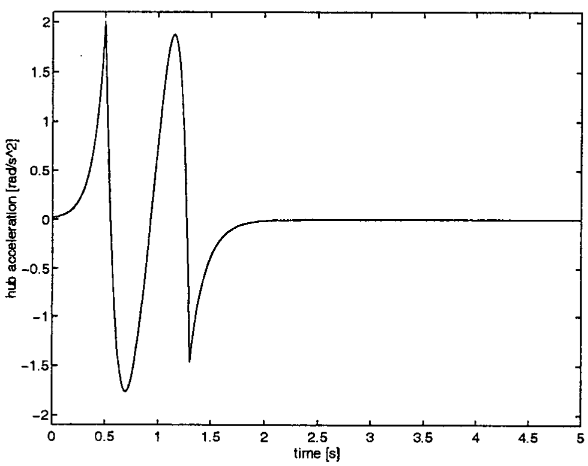


Fig. 5. Hub acceleration reference command input.

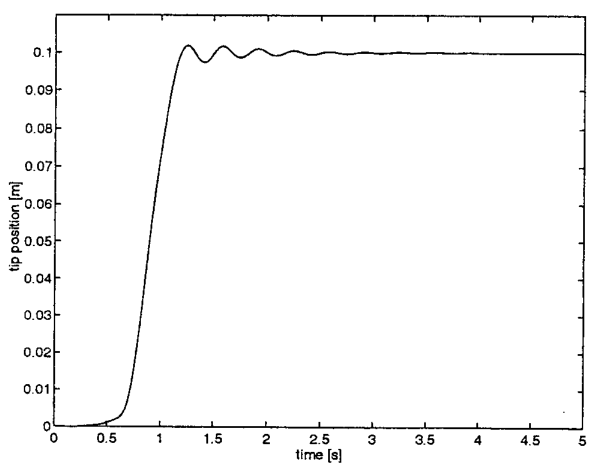


Fig. 6. Actual end-effector motion.

*IEEE International Conference on Robotics and Automation*, pp. 1102–1107, 1999.

[7] E. Gross, M. Tomizuka, "Experimental flexible beam tip tracking control with a truncated series approximation to uncancelable inverse dynamics", *IEEE Transactions on Control Systems Technology*, Vol. 2, No. 4, pp. 382–391, 1994.

[8] S. Panzieri, G. Ulivi, "Disturbance rejection of iterative learning control applied to trajectory tracking for a flexible link manipulator", *Proc. European Control*, pp. 2374–2379, 1995.

[9] H. A. Talebi, R. V. Patel, K. Khorasani, "Inverse dynamics control of flexible-link manipulators using neural networks", *Proc. IEEE International Conference on Robotics and Automation*, pp. 806–811, 1998.

[10] A. De Luca, S. Panzieri, G. Ulivi, "Stable inversion control for flexible link manipulators", *Proc. IEEE International Conference on Robotics and Automation*, pp. 799–805, 1998.

[11] A. Piazzi, A. Visioli, "Optimal noncausal set-point regulation of scalar systems", *Automatica*, Vol. 37, No. 1, pp. 121–127, 2001.

[12] A. Piazzi, A. Visioli, "Minimum-time system-inversion-based motion planning for residual vibration reduction", *IEEE/ASME Transactions on Mechatronics*, Vol. 5, No. 1, pp. 12–22, 2000.

[13] L. Marconi, C. Melchiorri, "The perfect tracking problem for nonminimum phase systems: applications to flexible-link robots", in *Modelling and control of mechanical systems*, A. Astolfi, D.J.N. Limebeer, C. Melchiorri, A. Tornambe and R.B. Vinter eds., Imperial College Press, London, pp. 169–185, 1997.

[14] L. Meirovitch, *Analytical methods in vibrations*, Macmillan, New York, 1967.

[15] L. Menini, A. Tornambe, L. Zaccarian, "Flexible robots", in *Modelling and control of mechanisms and robots*, C. Melchiorri and A. Tornambe eds, World Scientific, Singapore, pp. 115–148, 1996.

[16] D. Wang, M. Vidyasagar, "Transfer functions for a single flexible link", *Proc. IEEE International Conference on Robotics and Automation*, pp. 1042–1047, 1989.

[17] D. Wang, M. Vidyasagar, "Transfer functions for a single flexible link", *The International Journal of Robotics Research*, Vol. 10, No. 5, pp. 540–549, 1991.

[18] A. Piazzi, A. Visioli, "Minimum-time open-loop smooth control for point-to-point motion in vibratory systems", *Proc. IEEE International Conference on Robotics and Automation*, pp. 946–951, 1998.

[19] A. Piazzi, A. Visioli, "Robust set-point constrained regulation via dynamic inversion", *International Journal of Robust and Nonlinear Control*, to appear.

[20] H. Ratschek, J.Rokne, *New Computer Methods for Global Optimization*, Ellis Harwood Limited, Chichester (UK), 1988.

[21] G. Marro, A. Piazzi, "A geometric approach to multivariable perfect tracking", *Proc. of the 13th IFAC World Congress*, pp.241-246, 1996.

[22] G. Marro, "Multivariable regulation in geometric terms: old and new results", in *Colloquium on Automatic Control*, C. Bonivento, G. Marro and R. Zanasi eds, Springer-Verlag, New York, 1996.