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Brief Paper

Optimal noncausal set-point regulation of scalar systems☆

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Abstract

This paper presents a novel system inversion approach to the set-point regulation of linear scalar minimum-phase systems. This approach uses closed-form expressions of cause/effect pairs that make possible an arbitrarily smooth transition between two given output set-points. The cause/effect pairs are based on the introduced concept of "transition" polynomials. Simple optimization procedures are then proposed to solve the relevant optimal output synthesis and optimal input synthesis. The potentiality of the method is highlighted by the examples presented, namely, a synthesis of elevator velocity profiles and the performance improving of a PID controller. © 2000 Elsevier Science Ltd. All rights reserved.

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1. Introduction

In the control systems literature, the subject of output tracking has recently been approached with new methods based on system inversion procedures (DiBenedetto & Lucibello, 1993; Devasia, Chen & Paden, 1996; Devasia & Paden, 1998; Hunt, Meyer & Su, 1996; Hunt & Meyer, 1997). In many cases, once the desired output is known in advance, it is possible to perform a stable inversion, i.e. to determine a corresponding bounded (noncausal) input. As pointed out by Hunt and Meyer (1997), the actual control systems design can be centered on a feedforward/feedback scheme where the feedforward control is determined through stable inversion and a feedback regulator handles modeling and signal errors. The majority of the works pursuing this approach deals with nonlinear and nonminimum-phase systems and the emphasis is on algorithmic procedures to perform a stable inversion on a given output function.

In this paper, we propose a novel system inversion approach to the simplest, but fundamental, output tracking problem, i.e. set-point regulation of linear scalar minimum-phase systems. We will consider that the feedback regulator has already been properly designed by some method and concentrate on a given transfer function that represents the closed-loop or plant system (cf. Section 5.2). Then, unlike the previously cited papers, our emphasis is on synthesizing optimal input or output functions that are arbitrarily smooth and are to be used for the purpose of set-point constrained regulation. Consistently, our first aim is to determine, with closed-form expressions, a parameterized family of cause/effect pairs that permits an arbitrarily smooth transition, to be completed in (parameter) time τ, between two given output set-points (Section 2). The solution to this problem has its key in the introduced concept of "transition" polynomials that smoothly shape the system output (cf. expressions (9)–(11)). Secondly, two optimization problems are posed and solved: optimal output synthesis and optimal input synthesis (Sections 3 and 4, respectively). Specifically, in the set of cause/effect pairs the output transition time τ is, respectively, minimized, subjected to prespecified limits on the output derivatives and subjected to prespecified limits on the input and its derivatives. Overall, in the context of output tracking methodologies, the main contribution and the novelty of the paper is the introduction of the optimality issue in the inversion-based approach. For brevity, proofs of

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Propositions 2, 3, 6, 8, 9 and of Theorem 11 are reported in Piazzi and Visioli (2000b).

Notation. The set of integers is denoted by $\mathbb{N}$, whilst the sets of reals and positive reals are denoted by $\mathbb{R}$ and $\mathbb{R}^+$, respectively. We denote by P the space of the piecewise continuous scalar real functions and by $C^{(i)}$ the space of the scalar real functions which are continuous till the ith time derivative. Finally, D^i denotes the ith derivative operator.

2. Cause/effect pairs for set-point constrained regulation

Consider an nth-order scalar continuous linear system Σ with transfer function

$$G(s) := k_1 \frac{b(s)}{a(s)} = k_1 \frac{s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}. \quad (1)$$

The real rational $G(s)$ is stable and has minimum phase with all the zeros and poles belonging to the open left half plane. Moreover, $G(s)$ has no pole-zero cancellations: $a(s)$ and $b(s)$ are coprime. The input and output of Σ are $u \in \mathbb{R}$ and $y \in \mathbb{R}$, respectively. The relative order (or relative degree) of Σ is $\rho := n - m$. The subspace of external equilibrium points of Σ is denoted by

$$\mathscr{E} := \{(u, y) \in \mathbb{R}^2 \colon y = G(0)u\}. \quad (2)$$

The set of all cause/effect pairs associated with Σ (also known as the graph of Σ) is denoted by

$$\mathscr{B} := \{(u(\cdot), y(\cdot)) \in \mathrm{P} \times \mathrm{P} \colon D^n y + a_{n-1} D^{n-1} y + \cdots + a_0 y = k_1 (D^m u + b_{m-1} D^{m-1} u + \cdots + b_0 u)\}. \quad (3)$$

Suppose that system Σ is at the equilibrium point $e_a := (u_a, y_a) \in \mathscr{E}$ at the zero instant of the time axis. Roughly speaking, the addressed problems are:

(1) Find a sufficiently smooth cause/effect parametrized pair $(u(\cdot;\, \tau), y(\cdot;\, \tau)) \in \mathscr{B}$ with $\tau \in \mathbb{R}^+$ that permits the system transition to the new given equilibrium point $e_b := (u_b, y_b) \in \mathscr{E}$ under these requirements:
   (a) $y(t;\, \tau) = y_b \ \forall t \geq \tau$;
   (b) the image of $y(t;\, \tau)$ over $[0, \tau]$ is $[y_a, y_b]$, i.e. the time function $y(t;\, \tau)$ does not exhibit overshooting nor undershooting.

(2) Determine the minimum τ for the cases:
   (a) (optimal output synthesis) the absolute values of the derivatives of $y(t;\, \tau)$ till the kth order are within the given bounds:

$$|D^i y(t;\tau)| \leq y_M^{(i)} \quad \forall t \geq 0 \ \ i = 1, \ldots, k, \quad (4)$$

   (b) (optimal input synthesis) the absolute values of $u(t;\tau)$ and its derivatives till the lth order are within given bounds:

$$|D^i u(t;\, \tau)| \leq u_M^{(i)} \quad \forall t \geq 0 \ \ i = 0, 1, \ldots, l. \quad (5)$$

In setting the appropriate maximal order k and l of derivatives in (4) or (5) we may take advantage of this known property (cf. Polderman & Willems, 1998, p. 112).

Proposition 1. *Consider a pair* $(u(\cdot), y(\cdot)) \in \mathscr{B}$. *Then* $u(\cdot) \in C^{(l)}$ *if and only if* $y(\cdot) \in C^{(\rho + l)}$ *with* l *being a non-negative integer.*

A two-step procedure is proposed to solve Problem 1. First, with an arbitrarily high k, we synthesize $y(t;\, \tau) \in C^{(k)}$ that is a polynomial function in the time interval $[0, \tau]$ and satisfies the requirement (1b). Then, we compute $u(t;\, \tau) \in C^{(k-\rho)}$ by system inversion. In the following, without loss of generality, we can consider $G(0) = 1$ ($a_0 = k_1 b_0$), $e_a = (0, 0)$, and $e_b = (1, 1)$. In the interval $[0, \tau]$ define $y(t)$ as follows (here for simplicity we drop the parameter argument τ):

$$y(t) := c_0 + c_1 t + \cdots + c_{2k+1} t^{2k+1}. \quad (6)$$

The coefficient c_i can be determined by imposing the following continuity conditions that are necessary in order that $y(t)$ belong to $C^{(k)}$:

$$y(0) = 0, \quad Dy(0) = 0, \ \ldots, D^k y(0) = 0, \quad (7)$$

$$y(\tau) = 1, \quad Dy(\tau) = 0, \ \ldots, D^k y(\tau) = 0. \quad (8)$$

Exploiting the above conditions with definition (6), a system of $2k + 2$ linear equations with $2k + 2$ unknown coefficients arises.

Proposition 2. *The linear system of equations* (7), (8) *admits a unique solution for any* $\tau \in \mathbb{R}^+$ *and any* $k \in \mathbb{N}$.

The actual solution for $y(t;\, \tau)$ with $t \in [0, \tau]$ could be obtained by solving the system (7) (8) directly. A simpler alternative is to employ the closed-form expressions provided by the following proposition.

Proposition 3. *The unique solution of system* (7)–(8) *is given by*

$$y(t;\, \tau) = \frac{\int_0^t v^k (\tau - v)^k \, \mathrm{d}v}{\int_0^\tau v^k (\tau - v)^k \, \mathrm{d}v}, \quad t \in [0, \tau], \quad (9)$$

or, alternatively, by

$$y(t;\, \tau) = \frac{(2k+1)!}{(k!)^2 \tau^{2k+1}} \int_0^t v^k (\tau - v)^k \, \mathrm{d}v, \quad t \in [0, \tau]. \quad (10)$$

Moreover, $y(t;\, \tau)$ *over* $[0, \tau]$ *is monotonically increasing and its image is* $[0, 1]$.

By using the binomial formula to compute the integral appearing in (10) we obtain an explicit expression of the "transition" polynomial $y(t;\, \tau)$ that allows an arbitrarily smooth transition between two constant values (here 0 and 1):

$$y(t;\,\tau)=\begin{cases}0 \\ \quad \text{if } t\le 0,\\ \dfrac{(2k+1)!}{k!\tau^{2k+1}}\displaystyle\sum_{i=0}^{k}\frac{(-1)^{k-i}}{i!(k-i)!(2k-i+1)}\tau^i t^{2k-i+1} \\ \quad \text{if } 0\le t\le\tau,\\ 1 \quad \text{if } t\ge\tau.\end{cases} \quad (11)$$

Having determined $y(t;\,\tau)$ we can compute $u(t;\,\tau)$ by system inversion through Laplace transforms. Considering the assumption $G(0)=1$, we can rewrite the transfer function of Σ as

$$G(s)=\frac{\beta_m s^m+\beta_{m-1}s^{m-1}+\cdots+\beta_1 s+1}{\alpha_n s^n+\alpha_{n-1}s^{n-1}+\cdots+\alpha_1 s+1}.$$

The Laplace transform of $u(t;\,\tau)$ is then given by

$$U(s;\,\tau)=G^{-1}(s)Y(s;\,\tau) \quad (12)$$

with $Y(s;\,\tau)=\mathscr{L}[y(t;\,\tau)]$. By polynomial division we obtain

$$G^{-1}(s)=\gamma_\rho s^\rho+\gamma_{\rho-1}s^{\rho-1}+\cdots+\gamma_1 s+\gamma_0+H_0(s), \quad (13)$$

$$H_0(s)=\frac{\delta_{0,m-1}s^{m-1}+\delta_{0,m-2}s^{m-2}+\cdots+\delta_{0,0}}{\beta_m s^m+\beta_{m-1}s^{m-1}+\cdots+\beta_1 s+1}. \quad (14)$$

Here the strictly proper $H_0(s)$ represents the zero dynamics of Σ. Define $\eta_0(t):=\mathscr{L}^{-1}[H_0(s)]$ and then obtain the following result whose proof is a straightforward consequence of (12) and (13).

Proposition 4. *Consider $y(t;\,\tau)$ defined by* (11). *Provided that $k\ge\rho-1$*

$$u(t;\,\tau)=\gamma_\rho D^\rho y(t;\,\tau)+\gamma_{\rho-1}D^{\rho-1}y(t;\,\tau)+\cdots+\gamma_0 y(t;\,\tau) + \int_0^t \eta_0(t-v)y(v;\,\tau)\,\mathrm{d}v, \quad t\ge 0 \quad (15)$$

holds.

From (15), simplified expressions of $u(t;\,\tau)$ with $t\in[\tau,+\infty)$ are as follows:

$$u(t;\,\tau)=\gamma_0+\int_0^t \eta_0(t-v)y(v;\tau)\,\mathrm{d}v, \quad (16)$$

$$u(t;\,\tau)=\gamma_0+\int_0^\tau \eta_0(t-v)y(v;\tau)\,\mathrm{d}v+\int_\tau^t \eta_0(t-v)\,\mathrm{d}v. \quad (17)$$

By the virtue of minimum-phase assumption upon Σ, we have $u(t;\,\tau)$ bounded over $\mathbb{R}$. Moreover, it could be shown that $\lim_{t\to\infty}u(t;\,\tau)=1$.

Remark 5. Although other function basis could be used to solve Problem 1, the solution provided by (11) and (15) upon a polynomial base is computationally simple and suitable to be used in a variety of computer-based control applications (Piazzi & Visioli, 2000a).

3. Optimal output synthesis

In this section, we solve the optimal output synthesis problem (4), i.e. our aim is to determine

$$\tau^*=\min\{\tau>0\colon |D^i y(t;\,\tau)|\le y_M^{(i)}\ \forall t\ge 0\ i=1,\dots,k\}, \quad (18)$$

where $y(t;\,\tau)\in C^{(k)}$ is defined in (11) and $y_M^{(i)}$, $i=1,\dots,k$ are assigned bounds. The study of the derivatives of $y(t;\,\tau)$ is instrumental in solving (18).

Proposition 6. *Over the interval* $[0,\tau]$, *the derivatives of $y(t;\,\tau)$ can be expressed as* ($i=1,\dots,k+1$)

$$D^i y(t;\,\tau)=\frac{(2k+1)!}{(k!)^2\tau^{2k+1}}t^{k+1-i}(\tau-t)^{k+1-i}Q_{(i-1)}(t,\tau), \quad (19)$$

where $Q_{(i-1)}(t,\tau)$ denotes a suitable homogeneous polynomial of total order $i-1$ (defining $Q_{(0)}(t;\,\tau):=1$).

By virtue of Proposition 6, the homogeneous polynomial appearing in (19) can be defined according to ($i=0,1,\dots,k$)

$$Q_{(i)}(t,\tau)=\frac{(k!)^2\tau^{2k+1}}{(2k+1)!}\,\frac{D^{i+1}y(t;\,\tau)}{t^{k-i}(\tau-t)^{k-i}} \quad (20)$$

or, alternatively,

$$Q_{(i)}(t,\tau)=\frac{D^i[t^k(\tau-t)^k]}{t^{k-i}(\tau-t)^{k-i}}. \quad (21)$$

From direct inspection of $Dy(t;\,\tau)$ in (19), it is apparent that $Dy(\tau/2+t';\,\tau)$ is an even function of argument t'. Therefore, by well-known properties, $D^2y(\tau/2+t';\,\tau)$ is an odd function, $D^3y(\tau/2+t';\,\tau)$ an even function, $D^4y(\tau/2+t';\,\tau)$ an odd function, and so on. The main consequence of this alternating function sequence is that $|D^iy(\tau/2+t';\,\tau)|$ is an even t'-function for all $i\in\mathbb{N}$. Hence, optimization problem (18) is equivalent to

$$\min_{\tau>0}\tau \quad \text{s.t.} \quad \max_{[0,\,\tau/2]}|D^iy(t;\,\tau)|\le y_M^{(i)}, \quad i=1,\dots,k. \quad (22)$$

In solving the above optimization problem, it is useful to know the root distribution of polynomial $Q_{(i)}(t,\tau)$. The following lemma is a straightforward consequence of the homogeneousness of $Q_{(i)}(t,\tau)$.

Lemma 7. *Any root of $Q_{(i)}(t,\tau)$ can be expressed as $r\tau$ with r being a root of $Q_{(i)}(t,\,1)$.*

Proposition 8. *The roots of* $Q_{(i)}(t,1)$ *are distinct, real, and can be described as follows* ($i=1,\dots,k$):

$$r = 1/2 \pm \mu_{ij}, \quad j=1,\dots,p_i \ \text{ if } i \text{ is even}, \ (p_i := i/2), \tag{23}$$

$$r = 1/2,\ r = 1/2 \pm \mu_{ij}, j=1,\dots,p_i \ \text{ if } i \text{ is odd}, \quad (p_i := (i-1)/2), \tag{24}$$

$$0 < \mu_{i1} < \mu_{i2} < \cdots < \mu_{ip_i} < 1/2. \tag{25}$$

Proposition 9. *The global maximum of* $|D^i y(t;\tau)|$ *over* $[0,\tau/2]$ *can be determined as follows* ($i \le k$):

$$\max_{[0,\tau/2]} |D^i y(t;\tau)| = \max\{c_{k1}^{(i)}, c_{k2}^{(i)}, \dots, c_{kl_i}^{(i)}\}\frac{1}{\tau^i}, \tag{26}$$

the constants $c_{kj}^{(i)}$ *that do not depend on* τ *are accordingly defined as, if i is even* ($l_i := i/2$):

$$c_{kj}^{(i)} := |D^i y((1/2-\mu_{ij})\tau;\tau)|\tau^i, \quad j=1,2,\dots,l_i, \tag{27}$$

and if i is odd ($l_i := (i+1)/2$)

$$c_{k1}^{(i)} := |D^i y(1/2\tau;\tau)|\tau^i, \tag{28}$$

$$c_{kj}^{(i)} := |D^i y((1/2-\mu_{i,j-1})\tau;\tau)|\tau^i, \quad j=2,3,\dots,l_i. \tag{29}$$

By virtue of Proposition 9, optimization problem (22) is equivalent to

$$\begin{aligned} \min_{\tau>0} \tau \quad \text{s.t.} \quad & c_{k1}^{(1)}\frac{1}{\tau} \le y_M^{(1)}, \\ & c_{k1}^{(2)}\frac{1}{\tau^2} \le y_M^{(2)}, \\ & \max\{c_{k1}^{(3)}, c_{k2}^{(3)}\}\frac{1}{\tau^3} \le y_M^{(3)}, \\ & \dots \\ & \max\{c_{k1}^{(k)}, c_{k2}^{(k)}, \dots, c_{kl_k}^{(k)}\}\frac{1}{\tau^k} \le y_M^{(k)}. \end{aligned} \tag{30}$$

Problem (30) can be further transformed into

$$\begin{aligned} \min_{\tau>0} \tau \quad \text{s.t.} \quad & c_{k1}^{(1)}/y_M^{(1)} \le \tau, \\ & \sqrt{c_{k1}^{(2)}/y_M^{(2)}} \le \tau, \\ & \sqrt[3]{\max\{c_{k1}^{(3)}, c_{k2}^{(3)}\}/y_M^{(3)}} \le \tau, \\ & \dots \\ & \sqrt[k]{\max\{c_{k1}^{(k)}, c_{k2}^{(k)}, \dots, c_{kl_k}^{(k)}\}/y_M^{(k)}} \le \tau. \end{aligned} \tag{31}$$

The optimal τ^* is then given by the formula

$$\tau^* = \max\{c_{k1}^{(1)}/y_M^{(1)}, \sqrt{c_{k1}^{(2)}/y_M^{(2)}}, \dots, \sqrt[k]{\max\{c_{k1}^{(k)}, \dots, c_{kl_k}^{(k)}\}/y_M^{(k)}}\}. \tag{32}$$

4. Optimal input synthesis

In this section the problem addressed is on determining

$$\tau^{\#} := \min\{\tau > 0: |D^i u(t;\tau)| \le u_M^{(i)} \ \forall t \ge 0,\ i=0,1,\dots,l\}, \tag{33}$$

where $u_M^{(i)}$ are appropriate given bounds. First we provide an explicit procedure to compute the derivatives $D^i u(t;\tau)$. Assuming that $u(t;\tau) \in C^{(l)}$ with $u(t;\tau)=0$ for $t<0$, it follows that $\mathscr{L}[D^i u(t;\tau)] = s^i U(s;\tau)$ and, by the virtue of (12) and (13),

$$\mathscr{L}[D^i u(t;\tau)] = (\gamma_\rho s^{\rho+i} + \gamma_{\rho-1}s^{\rho+i-1} + \cdots + \gamma_0 s^i + s^i H_0(s))Y(s;\tau). \tag{34}$$

By polynomial division, the term $s^i H_0(s)$ can be rewritten according to ($i=0,1,\dots,l$)

$$s^i H_0(s) = \frac{\delta_{0,m-1}}{\beta_m}s^{i-1} + \frac{\delta_{1,m-1}}{\beta_m}s^{i-2} + \cdots + \frac{\delta_{i-2,m-1}}{\beta_m}s + \frac{\delta_{i-1,m-1}}{\beta_m} + H_i(s) \tag{35}$$

with $H_i(s)$, the ith-order zero-dynamics, defined as

$$H_i(s) = \frac{\delta_{i,m-1}s^{m-1} + \delta_{i,m-2}s^{m-2} + \cdots + \delta_{i,1}s + \delta_{i,0}}{\beta_m s^m + \beta_{m-1}s^{m-1} + \cdots + \beta_1 s + 1}. \tag{36}$$

Starting from $H_0(s)$, the computation of $H_i(s)$ can be done by recursion through

$$H_i(s) = sH_{i-1}(s) - \frac{\delta_{i-1,m-1}}{\beta_m}, \quad i=1,2,\dots,l \tag{37}$$

or, more explicitly, using

$$\delta_{i,0} = -\frac{\delta_{i-1,m-1}}{\beta_m}, \tag{38}$$

$$\delta_{i,j} = \delta_{i-1,j-1} - \delta_{i-1,m-1}\frac{\beta_j}{\beta_m}, \quad j=1,2,\dots,m-1. \tag{39}$$

Then, the following result is a straightforward consequence of (34) and (35).

Proposition 10. *Define* $\eta_i(t) := \mathscr{L}^{-1}[H_i(s)]$ *and the ith order derivative of* $u(t;\tau)$ *is given by* $i=1,2,\dots,l$

$$D^i u(t;\tau) = \sum_{j=0}^{\rho}\gamma_{\rho-j}D^{\rho+i-j}y(t;\tau) + \sum_{j=0}^{i-1}\frac{\delta_{j,m-1}}{\beta_m}D^{i-1-j}y(t;\tau) + \int_0^t \eta_i(t-v)y(v;\tau)\,dv. \tag{40}$$

The assumption of $u(t;\tau) \in C^{(l)}$ corresponds to considering $y(t;\tau) \in C^{(k)}$ with $k=\rho+l$ (cf. Proposition 1). Any procedure whose aim lies in solving (33) can rely on this theorem.

Theorem 11. *The feasible set relative to the constraint inequalities of problem* (33) *is not empty provided that* $u_M^{(0)} > 1$ and $u_M^{(i)} > 0,\ i = 1, 2, \ldots, l$.

Denote by $\mathscr{F}$ the feasible set of (33) according to

$$\mathscr{F} := \{\tau \in \mathbb{R}^+ : |D^i u(t;\tau)| \le u_M^{(i)}\ \forall t \ge 0\ i = 0, 1, \ldots, l\}.$$

A numerical approach to estimate $\tau^{\#}$ can be based on the following bisection-type algorithm ($\varepsilon \in \mathbb{R}^+$ is the given accuracy parameter).

Optimal Input Synthesis Algorithm

(1) $\tau^- = 0, \tau := 1$.
(2) Repeat $\tau := 2\tau$ until $\tau \in \mathscr{F}$.
(3) $\tau^+ := \tau$.
(4) While $\tau^+ - \tau^- > \varepsilon$
 (a) $\tau := (\tau^- + \tau^+)/2$.
 (b) If $\tau \in \mathscr{F}$ then $\tau^+ := \tau$ else $\tau^- := \tau$.
 End-while
(5) End.

Implementation of OISA requires checking $\tau \in \mathscr{F}$, i.e. solving a sequence of one-dimensional optimization problems. There are a variety of methods to do this, among them, the interval algorithms for global optimization (Hansen, 1992). Due to the structure of the feasible set $\mathscr{F}$ revealed in the proof of Theorem 1, OISA converges with certainty to a local minimum of Problem (33). Moreover, if $\mathscr{F} = [\tau^*, \infty)$, then the OISA converges to $\tau^{\#}$, i.e. at any stage of iterations $\tau^{\#} \in (\tau^-, \tau^+]$. In many instances we indeed have $\mathscr{F} = [\tau^*, \infty)$, but perhaps degenerate cases may exist for which $\mathscr{F} \ne [\tau^*, \infty)$. In order to secure the correct estimation of the global minimum $\tau^{\#}$, possible anomalous cases can be treated by coupling OISA with a gridding search on the real line or, alternatively, with the interval algorithms for tolerance optimization problems (Hansen, 1992).

5. Examples

5.1. Optimal synthesis of velocity profiles for elevators

As an example of optimal output synthesis we determine the optimal velocity function of an elevator that starts moving with velocity equal to zero and has to reach in minimum time the cruise velocity v_M with constraints on the maximum acceleration a_M and on the maximum jerk j_M, in order to assure a comfortable travel to the passengers. Choosing $y(t;\tau) \in C^{(2)}$ $(k = 2)$ so that the jerk function is continuous, we have that the velocity function is, according to (11):

$$v(t;\tau) = v_M y(t;\tau) = v_M\left(\frac{6}{\tau^5}t^5 - \frac{15}{\tau^4}t^4 + \frac{10}{\tau^3}t^3\right)$$

for $t \in [0, \tau]$ and $v(t;\tau) = v_M$ for $t > \tau$. Hence, the above problem can be formulated as follows:

$$\min_{\tau \in \mathbb{R}^+} \tau \quad \text{s.t.} \quad |v_M D y(t;\tau)| \le a_M,\ |v_M D^2 y(t;\tau)| \le j_M \quad \forall t \ge 0. \tag{41}$$

Fixing $v_M = 10$ m/s, $a_M = 2$ m/s^2 and $j_M = 0.5$ m/s^3, problem (41) is equivalent to

$$\tau^* = \min\{\tau > 0 : |Dy(t;\tau)| \le y_M^{(1)},\ |D^2 y(t;\tau)| \le y_M^{(2)}\ \forall t \in [0, \tau]\}, \tag{42}$$

where τ^* is the sought optimal transition time and $y_M^{(1)} = 0.2$, $y_M^{(2)} = 0.05$. For $k = 2$, the solution (32) specializes

$$\tau^* = \max\{c_{k1}^{(1)}/y_M^{(1)}, \sqrt{c_{k1}^{(2)}/y_M^{(2)}}\}, \tag{43}$$

where $c_{k1}^{(1)} = 15/8$ and $c_{k1}^{(2)} = 5.7735$ are determined according to (28) and (27), respectively. Hence $\tau^* = 10.75$ s results. The resulting velocity, acceleration and jerk functions are plotted in Fig. 1. It appears how the active constraint of problem (41) is the one on the jerk function, which reaches the jerk bound twice during the transient, and how the velocity function is monotonically increasing, according to Proposition 3.

5.2. Improving the performances of a PID controller

Consider a plant, with transfer function

$$P(s) = \frac{377(s+2)}{[(s+2)^2 + 9][(s+3)^2 + 49]},$$

that is subjected to the output regulation of a PID controller using a typical unity-feedback system. The PID

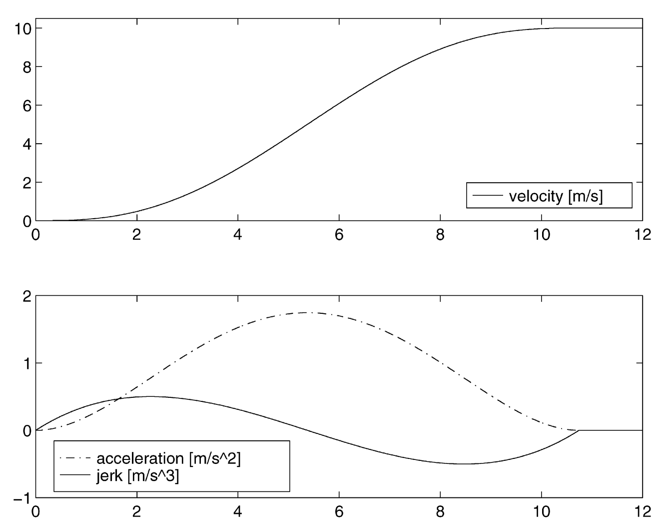

Fig. 1. The resulting velocity, acceleration and jerk functions of the elevator example.

controller has the (proper) structure

$$C(s) = K_c \frac{s^2 + 2\delta\omega_n s + \omega_n^2}{s(s+50)},$$

where the PID tuning parameters K_c, δ and ω_n have been chosen, in a classic control setup, to minimize the settling time of the unit-step response subjected to (i) a stable closed-loop; (ii) an overshooting not exceeding S% of the steady-state output; and (iii) a control variable $u(t)$ not exceeding the saturation limit u_{SAT}. For the case at hand, $S = 10$ and $u_{SAT} = 3$, the optimal values of the PID parameters have been determined by means of genetic algorithms: it results $K_c = 7.6172$, $\delta = 0.4323$ and $\omega_n = 5.1073$ rad/s. The obtained settling time is equal to 1.68 s (here defined as the time it takes for the output to remain into a range of 2% of the steady-state value). Plots of the control variable u and of the system output y are shown in Fig. 2.

The set-point regulation performance of the above control scheme can be significantly improved by substituting the unit-step reference with a new reference function designed by means of the optimal input synthesis described in Section 4. First, a cause/effect pair $(u(t;\tau); y(t;\tau)) \in \mathscr{B}$ associated with $P(s)$ has to be determined such as $u(t;\tau) \in C^{(0)}$. The relative order of $P(s)$ is $\rho = 3$ so that, by Proposition 1, $y(t;\tau) \in C^{(3)}$. Hence, according to (11),

$$y(t;\tau) = -20\frac{t^7}{\tau^7} + 70\frac{t^6}{\tau^6} - 84\frac{t^5}{\tau^5} + 35\frac{t^4}{\tau^4}, \quad t \in [0,\tau].$$

Then, the OISA has to be applied to solve the following optimization problem:

$$\min_{\tau \in \mathbb{R}^+} \tau \quad \text{s.t.} \quad |u(t;\tau)| \le u_{SAT} \quad \forall t \ge 0.$$

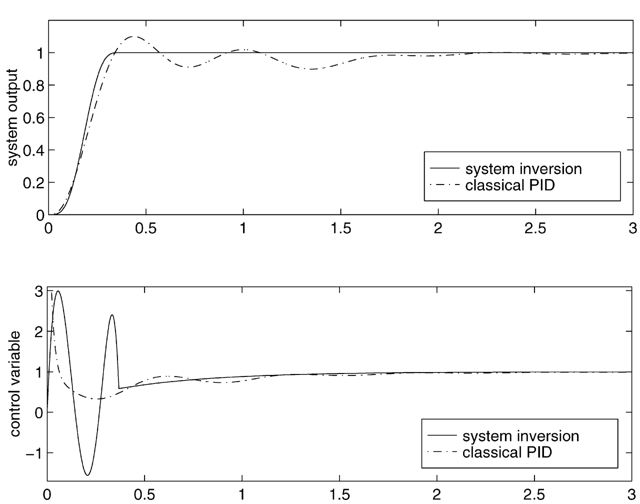

Fig. 2. Control variable u and system output y with the classical PID scheme and with the inversion-based approach.

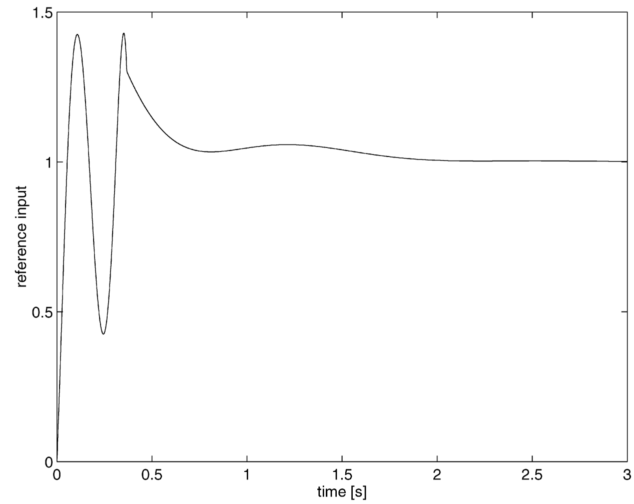

Fig. 3. Optimal reference input signal with the inversion-based approach.

It results in the minimum transition time $\tau^{\#} = 0.367$ s. Having determined the system output $y(t;\tau^{\#})$, the system inversion can be applied to the closed-loop system

$$T(s) = \frac{C(s)P(s)}{1 + C(s)P(s)}$$

in order to determine the optimal reference input

$$r(t;\tau^{\#}) = \mathscr{L}^{-1}[T^{-1}(s)Y(s;\tau^{\#})].$$

The resulting optimal reference input, determined with Proposition 6, is plotted in Fig. 3, whilst the corresponding control variable and system output are shown in Fig. 2. It is worth noticing that the control variable does not exceed the saturation level, as expected. The obtained settling time is equal to 0.31 s, which is more than five times less than the one achieved with the unit-step reference.

6. Conclusions

For linear scalar systems, this paper has presented a noncausal technique that provides the synthesis of an optimal open-loop (feedforward) control signal to be used for the purpose of set-point constrained regulation, in which smoothness constraints are imposed on the plant input or output functions. This technique implicitly assumes that a feedback regulator has previously been designed independently from the optimal feedforward command signal. Indeed, as it has been shown in the examples, the use of an optimal noncausal command input instead of the typical step-input can considerably improve the set-point regulation performances of a given control scheme. As a consequence, applying this inversion based method is straightforward and the potential usefulness of the technique appears significant in the control engineering field, especially for the motion control problems. In particular, for the point-to-point motion in mechanical systems the methodology has been proven effective and relatively unaffected by parameter variations even if applied as a plain open-loop strategy (Piazzi & Visioli, 2000a). However, from a robustness viewpoint the best results can be obtained with a combined synthesis of the noncausal feedforward action with the feedback regulator (Piazzi & Visioli, 1998).

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