Technical Notes and Correspondence

A New Solution to the Regulator Problem with Output Stability

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Abstract—**A new solution to the regulator problem with output stability (external disturbance is also considered) is achieved by straightforward application of self-hidden conditioned invariants. The proposed solution, also, has an interesting feature: in the corresponding induced synthesis it is always possible to avoid hard eigenspace computations. A possible order reduction in the compensator synthesis is discussed.**

I. INTRODUCTION

The main idea behind this note for solving a well-known problem in the geometric approach, i.e., the regulator problem with output stability, concerns the use of the concept of self-hidden conditioned invariance. This concept is the dual of self-bounded controlled invariance which was introduced by Basile and Marro [1] in order to achieve straightforward treatment of constrained controllability. Further studies indicate that these concepts can be very useful in approaching many problems in the linear system theory, especially when stability requirement is considered [2]-[4]. An interesting feature of the use of self-hidden and self-bounded invariants is the possibility of avoiding eigenspace computations: the basic algorithms of the geometric approach (i.e., determination of the maximum controlled invariant contained in a given subspace or/and its dual; refer to the fundamental work [6] for details) are sufficient.

II. SOME NOTATION AND GENERAL BACKGROUND

Consider the following equations:

$$\dot{x}(t)=Ax(t)+Bu(t)+Dd(t), \tag{1a}$$

$$y(t)=Cx(t), \tag{1b}$$

$$e(t)=Ex(t); \tag{1c}$$

where $x \in \mathcal{R}^n$ denotes the system state vector, $u \in \mathcal{R}^p$ the controlled input, $d \in \mathcal{R}^r$ the disturbance input, $y \in \mathcal{R}^q$ the observed output, and $e \in \mathcal{R}^s$ the regulated output. Let $\mathcal{B} := \text{Im } B$, $\mathcal{D} := \text{Im}, D$, $\mathcal{C} := \ker C$, $\mathcal{E} := \ker E$.

A generic A-invariant, $(A, \mathcal{B})$-controlled invariant and $(A, \mathcal{C})$-conditioned invariant will be indicated by $\mathcal{J}$, $\mathcal{V}$, $\mathcal{S}$, respectively. We denote with $\mathcal{K}$ and $\mathcal{G}$, respectively, the infimum and the supremum of the sets $\Psi = \{\mathcal{S}: \mathcal{S} \supseteq \mathcal{D}\}$, $\Phi = \{\mathcal{V}: \mathcal{V} \subseteq \mathcal{E}\}$ (i.e., the minimum $(A, \mathcal{C})$-conditioned invariant containing $\mathcal{D}$ and the maximum $(A, \mathcal{B})$-controlled invariant contained in $\mathcal{E}$). For the reader's convenience, we report the following definition.

Definition: An $(A, \mathcal{C})$-conditioned invariant $\mathcal{S}$, containing a given subspace $\mathcal{D}$, is said to be *self-hidden* relative to $\mathcal{D}$ if $\mathcal{S} \subseteq \mathcal{K} + \mathcal{C}$.

The set of all self-hidden $(A, \mathcal{C})$-conditioned invariants relative to $\mathcal{D}$ (i.e., a subset of Ψ) has the property of being a lattice with respect to intersection and sum. This fact leads to several further properties especially in relation to stabilizability, as has been shown by Basile and Marro in [4]. Let $\Psi_1 = \{\mathcal{S} \in \Psi : \mathcal{S} \subseteq \mathcal{K} + \mathcal{C}, \mathcal{S} \subseteq \mathcal{E}\}$ be the subset of self-hidden $(A, \mathcal{C})$-conditioned invariants relative to $\mathcal{D}$, and contained in $\mathcal{E}$; and $\mathcal{L}$ be its supremum. The main link between self-hiddeness and external stabilizability of conditioned invariants is the following.

Lemma: If $\mathcal{L}$, the maximum self-hidden $(A, \mathcal{C})$-conditioned invariant relative to $\mathcal{D}$ and contained in $\mathcal{E}$, is not externally-stabilizable, no other externally-stabilizable $(A, \mathcal{C})$-conditioned invariant containing $\mathcal{D}$ and contained in $\mathcal{E}$ exists.

The dual of Lemma 1 was conjectured in [1] and proved in [7] and [2]; it refers to the stabilizability property of the minimum self-bounded $(A, \mathcal{B})$-controlled invariant relative to $\mathcal{E}$ and containing $\mathcal{D}$.

It was proved in [5] that the following relation holds:

$$\mathcal{L}=\mathcal{K}+\mathcal{N}, \tag{2}$$

where $\mathcal{N}$ is the maximum $(A, \mathcal{D})$-controlled invariant contained in $\mathcal{C} \cap \mathcal{E}$. Note that the determination of $\mathcal{L}$ is based, as claimed in the Introduction, on the fundamental algorithms of the geometric approach.

III. THE NEW RESULT

The set of equations in (1) describes the controlled system that has to be regulated; as regulator let us consider a dynamic system modeled by the equations:

$$\dot{z}(t)=Nz(t)+My(t), \tag{3a}$$

$$u(t)=Lz(t)+Ky(t). \tag{3b}$$

The overall system is shown in Fig. 1. We deal with the problem.

Problem (Regulator Problem with Output Stability): Find, if possible, a dynamic regulator such that:

i) $e(t)=0, \quad t\geq 0$

for any piecewise continuous $d(t)$, when $x(0) = 0$, $z(0) = 0$;

ii) $\lim_{t\to\infty} e(t)=0$

for any $x(0)$ and $z(0)$ when $d(t) = 0$, $t \geq 0$.

Condition i) is the *structure requirement* and ii) is the *output stability requirement*. We do not assume any specific hypothesis on the controlled system, for example, stabilizability or detectability; the controlled system can be regarded as the aggregation of an *endosystem* (i.e., the plant) and an *exosystem*. The exosystem reproduces, for example, signals to be tracked or "smooth disturbance." The following theorem was presented in [8] by Schumacher.

Theorem 1: The regulator problem with output stability has a solution if and only if there exist an $(A, \mathcal{C})$-conditioned invariant $\mathcal{S}$ and an $(A, \mathcal{B})$-controlled invariant $\mathcal{V}$ satisfying the conditions:

$$\mathcal{D} \subseteq \mathcal{S} \subseteq \mathcal{V} \subseteq \mathcal{E}, \tag{4a}$$

$$\mathcal{S} \text{ is externally-stabilizable}, \tag{4b}$$

$$\mathcal{V} \text{ is externally-stabilizable}. \tag{4c}$$

From Theorem 1 it follows immediately that the problem has a solution if and only if $\mathcal{G}$ is externally-stabilizable and $\mathcal{G} \supseteq \mathcal{S}$. $\mathcal{S}$ denotes the infimum of the set

$$\Psi_2=\{\mathcal{S} \in \Psi : \mathcal{S} \text{ externally-stabilizable}\},$$

i.e., the minimum externally-stabilizable $(A, \mathcal{C})$-conditioned invariant containing $\mathcal{D}$.

By means of this easy result, it is clear that a synthesis procedure based

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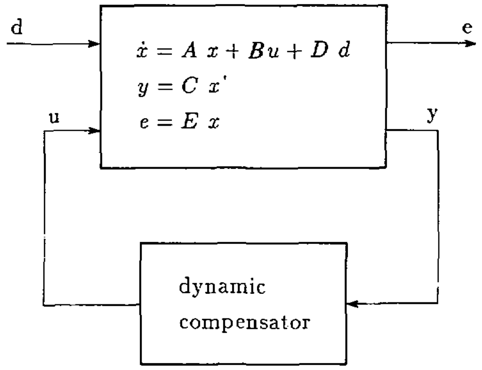

Fig. 1. The controlled system and the regulator.

upon $\mathcal{G}$ and $\mathcal{J}$ is available; the synthesized regulator will have order dimension equal to $n - \dim(\mathcal{G})$. Note that the choice of the above subspaces is only a possible peculiar choice among many subspaces that may satisfy Theorem 1 requirements. Another, more convenient choice will be available through the following new result.

Theorem 2: The regulator problem with output stability has a solution if and only if:

$$\mathcal{K} \subseteq \mathcal{J}, \tag{6a}$$

$$\mathcal{J} \text{ is externally-stabilizable,} \tag{6b}$$

$$\mathcal{L} \text{ is externally-stabilizable.} \tag{6c}$$

Proof:

If part: By relations (2) and (6a) we deduce $\mathcal{L} \subseteq \mathcal{J}$; so that these subspaces in virtue of conditions (6b) and (6c) satisfy all the requirements of Theorem 1.

Only if part: There exist an externally-stabilizable $(A, \mathcal{C})$-conditioned invariant $\mathcal{S}$ and an externally-stabilizable $(A, \mathcal{B})$-controlled invariant $\mathcal{V}$ that satisfy relations (4a). First we obtain condition (6a) through relations $\mathcal{K} \subseteq \mathcal{S}$ and $\mathcal{V} \subseteq \mathcal{J}$. By virtue of the latter relation, the external stabilizability property of $\mathcal{V}$ implies that $\mathcal{J}$ is also externally-stabilizable as an $(A, \mathcal{B})$-controlled invariant. Finally, the last condition (6c) is obtained as a consequence of the presented Lemma.

As we have just seen the proof of Theorem 2 is very simple; on the other hand if we have $\mathcal{D} = \mathcal{O}$ (i.e., the controlled system is not subject to external disturbance) we recover the "extended regulator problem" (see [9]) and obtain the following result as a corollary of Theorem 2.

Theorem 3: Suppose that $D = 0$. The regulator problem with output stability has a solution if and only if:

$$\mathcal{J} \text{ is externally stabilizable,} \tag{7a}$$

$$\mathcal{M} \text{ is externally stabilizable as } (A, \mathcal{C}) \text{ conditioned invariant,} \tag{7b}$$

where $\mathcal{M}$ is the supremum of the set $\Gamma = \{\mathcal{J} : \mathcal{J} \subseteq \mathcal{C} \cap \mathcal{E}\}$.

Proof: We specify Theorem 2 conditions in the case $D = 0$. First note that condition (6a) degenerates being $\mathcal{K}|_{\mathcal{D}=\mathcal{O}} = \mathcal{O}$. Condition (7a) is the same as that in (6b). From relation (2) we obtain $\mathcal{L}'|_{\mathcal{D}=\mathcal{O}} = \mathcal{N}|_{\mathcal{D}=\mathcal{O}}$, but $\mathcal{N}|_{\mathcal{D}=\mathcal{O}} = \mathcal{M}$ so by virtue of (6c) $\mathcal{M}$ as $(A, \mathcal{C})$-conditioned invariant is externally-stabilizable.

IV. Conclusion

As claimed in the Introduction, we have substituted the hard computation of $\mathcal{G}$ by the easier computation of $\mathcal{L}$ and a simple stabilizability check. Since in general the relation $\mathcal{G} \subseteq \mathcal{L}$ holds, it follows that:

$$\dim(\mathcal{G}) \le \dim(\mathcal{L}),$$

and an order reduction in a actual synthesis procedure may be obtained.

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Derivation of Continued-Fraction Expansions from the Coefficients of Power-Series Expansions of a Transfer Function

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Abstract—**This correspondence presents a rule for deriving various forms of continued-fraction expansions of a transfer function from its power-series expansions around several points. This rule is especially useful for computing multipoint Padé approximants to a given function rational or irrational.**

I. Introduction

Continued-fraction expansions (CFE's) have been widely used in mathematics, physics, and engineering [1]-[3]. One of the most important applications in the simulation and design of control systems is to obtain reduced-order models for linear time-invariant systems. Research into this subject has been quite active [5]-[13] since it was first suggested by Chen and Shieh [4]. The continued-fraction approach to the reduction problem of a given system described by its state-space model, in general, requires that the corresponding transfer function be first determined. However, it would be cumbersome and quite unsuitable for numerical work when the order of the system is high. Thus, it is desirable to have an algorithm, without knowing the transfer function, for determining the CFE directly from the (weighted) time-moments and/or Markov parameters of the system, i.e., the coefficients of the power-series expansions around $s = 0$ and/or $s = \infty$ of the transfer function. Many efforts have been devoted to this direction [14]-[24].

The relationship between the power-series expansion and the single-point CFE of a function has been exhibited [1]. Recently, it has been shown [25]-[27] that reduced-order models obtained by the CFE method are the same as those by Padé approximation obtained from power-series expansions. Notice, however, that computational economy is gained by using continued-fraction approach rather than by using the Padé technique [28], the latter involving the inversion of a Toeplitz matrix.

The purpose of this correspondence is to present a rule for obtaining multipoint CFE's of a transfer function from its power-series expansions.

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