HERMITE POLYNOMIALS FOR ITERATIVE OUTPUT REPLANNING FOR FLAT SYSTEMS AFFECTED BY ADDITIVE NOISE

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ABSTRACT

The paper considers the output tracking problem for nonlinear systems whose performance output is also a flat output of the system itself. A desired output signal is sought on the actual performance output by using a feedforward inverse input that is periodically updated with discrete-time feedback of the sampled state of the system. The proposed method is based on an iterative output replanning that uses the desired output trajectory and the sampled state to replan an output trajectory whose inverse input helps in reducing the tracking error. This iterative replanning exploits the Hermite interpolating polynomials to achieve an overall arbitrarily smooth input and a tracking error that can be made arbitrarily small if the state sampling period is sufficiently small and mild assumptions are considered. Some simulation results are presented for the cases of a unicycle and a one-trailer system affected by additive noise.

Key Words: flat systems, output tracking, nonholonomic vehicles.

I. INTRODUCTION

A fundamental problem in control theory for automation is output tracking [1]. Given a desired signal or reference on the output variable of a controlled dynamical system, the problem is to appropriately manipulate the input of the system in such a way that the actual output follows the desired reference as closely as possible. The classic solution approach prescribes the design of a feedback controller that can asymptotically zero the tracking error [2–6]. When the desired output is known in advance, an alternative tracking control strategy is inversion-based control. It is a feedforward/feedback strategy where the feedforward is determined by stable input-output inversion and the feedback is activated by a controller whose input is the error between the reference state and the actual state [7–9]. A variant of this strategy considers the application of a feedback controller first to reduce the effects of unmodeled dynamics or uncertainties on the controlled system and then the closed-loop dynamic is inverted by the stable inversion procedure [10].

Both the classic approach to output tracking and the newer inversion-based approach expect the continuous-time availability of the measured output or the measured state of the controlled system. However, there are cases where continuous-time or high-frequency revelation of the system state or output is not possible or not economical and only low-frequency feedback is practicable. The resulting control framework is then a hybrid feedforward/feedback scheme where the controlled system is commanded by feedforward (*i.e.* open-loop) inputs that are periodically updated to compensate or reduce the tracking error.

This paradigm has been pioneered in [11,12] for the robust stabilization of nonlinear driftless and chained systems; an application was also proposed for the lane following control of a vision-based autonomous car [13].

In this paper, within this framework of hybrid feedforward/feedback control, we propose the output tracking of a nonlinear flat system affected by additive noise on its state derivative. More specifically, we consider a controlled system whose performance output is a flat output of the system itself [14,15]. A desired output signal is sought on the actual output by using a feedforward inverse input that is periodically updated using the observation of the full system state acquired at intervals of period T . The proposed method is actually an iterative output replanning that uses the desired output trajectory and the updated state to replan an output trajectory whose inverse input helps in reducing the tracking error. This iterative replanning exploits the Hermite interpolating polynomials to achieve an overall arbitrarily smooth input and a tracking error that can be made arbitrarily small under mild assumptions if the sampling period T is sufficiently small.

Paper contribution. The main contribution of the paper is the development of a new method for output tracking based on discrete-time output measurements. The advantages of the method are its generality (it can be applied to a wide class of flat systems) and the fact that it allows designing control signals with an arbitrary degree of continuity.

This paper is a revised and extended version of our conference paper [16].

Notation. If x is a $C^l(\mathbb{R}, \mathbb{R}^n)$ function, $x^{(l)}$ denotes the derivative of x of order l , $x(t_0^+)$ and $x(t_0^-)$ denote, respectively, $\lim_{x \rightarrow t_0^+} x(t)$ and $\lim_{x \rightarrow t_0^-} x(t)$. For any vector $v \in \mathbb{R}^n$, $(v)_i$ denotes its i -th component.

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II. PROBLEM STATEMENT

Consider the nonlinear controlled system

$$\dot{x} = f(x, u), \quad (1)$$

with $x \in C(\mathbb{R}, \mathbb{R}^n)$, $u \in C(\mathbb{R}, \mathbb{R}^m)$. System (1) is flat if there exists an output function y such that the system state $x(t)$ and the input $u(t)$ can be written as a function of y and its derivatives up to a finite order, evaluated at time t . More precisely the following definition can be given (see [17]).

Definition 1. System (1) is flat if there exist a flat output y of dimension m , two integers r and s and mappings ψ from $\mathbb{R}^n \times \mathbb{R}^{m(s+1)}$ to $\mathbb{R}^m$, of rank m in a suitably chosen output subset, and (ϕ_0, ϕ_1) from $\mathbb{R}^{m(r+2)}$ to $\mathbb{R}^n \times \mathbb{R}^m$, of rank $m+n$ in a suitable open subset, such that

$$y = (y_1, \dots, y_m) = \psi(x, u, \dot{u}, \dots, u^{(s)}) \quad (2)$$

implies that

$$\begin{aligned} x &= \phi_0(y, \dot{y}, \dots, y^{(r)}), \\ u &= \phi_1(y, \dot{y}, \dots, y^{(r+1)}), \end{aligned} \quad (3)$$

the differential equation $\frac{d\phi_0}{dt} = f(\phi_0, \phi_1)$ being identically satisfied.

In this way, function ϕ_0 represents the state x with the output y and its derivatives up to the order r . Function ϕ_1 represents the input u with the output and its derivatives up to the order $r+1$. For simplicity, for a C^n function f we use the notation $\bar{f}^n = (f, f^{(1)}, \dots, f^{(n)})$, to denote the ordered set containing function f and its time derivatives up to the order n . If ϕ_1 is sufficiently regular, differentiating (3), one obtains functions ϕ_i , such that, for any $i \geq 1$

$$u^{(i-1)} = \phi_i(\bar{y}^{r+i}), \quad (4)$$

i.e., the input derivatives can be expressed as a function of the output and its derivatives. Similarly, if ψ is sufficiently regular, differentiating (2), one obtains functions ψ_i , such that, for any $i \geq 0$

$$y^{(i)} = \psi_i(x, \bar{u}^{s+i}), \quad (5)$$

(with $\psi_0 = \psi$, where ψ is given in (2)). Combining (4) and (5), the following identity holds $\forall i \geq 1$

$$u^{(i-1)} = \phi_i(\psi_0(x, \bar{u}^s), \dots, \psi_{r+i}(x, \bar{u}^{(s+r+i)})). \quad (6)$$

It is well known that tracking and motion planning problems can be easily solved for flat systems, see for instance chapter 7 of [17]. In this paper, we study the tracking problem for system (1), in the presence of a bounded disturbance added to the nominal velocity of the state:

$$\dot{x}(t) = f(x(t), u(t)) + \eta(t), \quad (7)$$

where η is a disturbance signal such that $\|\eta(t)\| \leq N$, $\forall t \in \mathbb{R}$. The performance output of system (7) is given by

$$y = \psi(x, u). \quad (8)$$

We assume that y is a flat output for system (7) when no noise is present (i.e., $\eta = 0$). In this case, from (2) and (3), it follows that the output signal y satisfies

$$y = \psi(\phi_0(\bar{y}^r), \phi_1(\bar{y}^{r+1})). \quad (9)$$

Note that the form (7) may be restrictive since the disturbance η enters as a pure additive term. This form does not include, for instance, cases in which a disturbance multiplies the state x or the input u .

We assume that the full system state is acquired periodically, with a sampling period equal to $T > 0$. In this way, the feedback control relies on the discrete-time observed sequence $x(kT)$, $k \in \mathbb{N}$. For instance, this assumption is reasonable when the system state is obtained through a camera, using computer vision techniques. In this case, a sampling time of $T = 0.1$ seconds would be a typical situation.

In this paper we study an iterative output replanning technique for controlling system (7), based on Hermite interpolating polynomials, similar in spirit to the iterative state steering method presented in [18]. Roughly speaking, the method is as follows. A sufficiently regular reference output trajectory y_d is assigned in advance. During each time interval $[kT, (k+1)T]$, a replanned output y_p is computed such that:

1. y_p corresponds through (3) to an initial state which is the same as $x(kT)$, i.e. $x(kT) = \phi_0(y_p(kT), \dot{y}_p(kT), \dots, y_p^{(r)}(kT))$;
2. the replanned output y_p converges to the desired one y_d at time $(k+1)T$, i.e., $y_p((k+1)T) = y_d((k+1)T)$.

The control is given according to (3), $\forall t \in [kT, (k+1)T]$, by

$$u(t) = \phi_1(y_p(t), \dot{y}_p(t), \dots, y_p^{(r+1)}(t)).$$

Since the system is affected by additive noise and in interval $[kT, (k+1)T]$ open loop control is used, at time $(k+1)T$ the system output $y((k+1)T)$ is different from $y_d((k+1)T)$. Hence, the above step is repeated, finding a new replanned trajectory y_p , that would drive the output of the nominal system to y_d at time $(k+2)T$. Again, for the presence of noise, at time $(k+2)T$ the actual system output is different from the reference trajectory and a new trajectory is replanned. Roughly speaking, since the replanned trajectories converge to the reference y_d , the system output is driven towards the desired output and the tracking error is kept limited despite the presence of a disturbance. This method is illustrated in Fig. 1, while Fig. 2 shows the corresponding control scheme. In the paper, we prove that the tracking error can be made arbitrarily small if the replanning time T is chosen sufficiently small. Moreover, we show that the replanned output

Fig. 1. The iterative replanning method. The figure shows the reference output trajectory y_d , the actual system output y and the replanned trajectory y_p .

Fig. 2. The iterative control scheme for the trajectory tracking of a flat system.

y_p can be chosen in such a way to have an arbitrary degree of continuity on the resulting input function.

III. THE METHOD OF ITERATIVE REPLANNING

3.1 An Hermite interpolation problem

In this section we consider the following problem.

Replanning problem. Given flat system (1), an output reference trajectory, $y_d \in C^{r+l}(\mathbb{R}, \mathbb{R}^m)$, an initial state $x_0 \in \mathbb{R}^n$ and initial values for the input and its derivatives $u_0, u_0^{(1)}, \dots, u_0^{(l-1)}$, find an output reference trajectory $y_p \in C^{r+l}(\mathbb{R}, \mathbb{R}^m)$ such that the following properties hold:

1. $\phi_0(\bar{y}_p'(0)) = x_0$, i.e., x_0 is the initial state of the system trajectory that has y_p as output,
2. $u_0^{(i)} = \phi_{i+1}(\bar{y}_p^{r+i+1}(0))$, $i = 0, \dots, l-1$ i.e., $u_0^{(i)}$ is the initial value of the i -th derivative of the control for the system trajectory which has y_p as output,
3. $y_p(t) = y_d(t)$, $\forall t \geq T$, where T is a given positive constant, i.e. function y_p converges to y_d at time T .

For any $l \in \mathbb{N}$, let $\Psi_0, \Psi_1, \dots, \Psi_{r+l}$ be vectors in $\mathbb{R}^m$ and set matrix $\Psi = (\Psi_0, \Psi_1, \dots, \Psi_{r+l})$. Consider the interpolation problem of determining a function $\pi_{\Psi,T} \in C^{r+l}([0, T], \mathbb{R}^m)$ that satisfies the two conditions

$$\frac{d^i}{dt^i} \pi_{\Psi,T}(0) = \Psi_i, \quad i = 0, \dots, r+l, \quad (10)$$

$$\frac{d^i}{dt^i} \pi_{\Psi,T}(T) = 0, \quad i = 0, \dots, r+l. \quad (11)$$

Fig. 3. The first three Hermite polynomials for $r+l=4$ and $T=1$.

Condition (10) requires that function $\pi_{\Psi,T}$ have the first $r+l$ derivatives equal to the columns of Ψ at time $t=0$, while condition (11) requires that all derivatives up to the $(r+l)$ -th be equal to 0 at time $t=T$.

This problem belongs to the class of Hermite interpolation problems, which have been widely studied in interpolation literature. Its solution can be written in the form:

$$(\pi_{\Psi,T})_i(t) = \sum_{k=0}^{r+l} A_{T,k}(t) (\Psi_k)_i, \quad (12)$$

where the Hermite interpolation function $A_{T,k}$ is the minimum degree polynomial that satisfies conditions $\frac{d^i}{dt^i} A_{T,k}(0) = \delta_{i-k}$, $\frac{d^i}{dt^i} A_{T,k}(T) = 0$, where $\delta_0 = 1$ and $\delta_i = 0$ for $i \neq 0$. These polynomials have degree $2(r+l+1)$ and can be computed in closed form using a result presented in [19]:

$$A_{T,k}(t) = (t-T)^{r+l+1} \frac{t^k}{k!} \sum_{i=0}^{r-k+l} \frac{(-t)^i}{i!} \frac{(r+i+l)!}{(-T)^{r+l+i+1} (r+l)!}. \quad (13)$$

These polynomials satisfy the following inequality, $\forall t \in [0, T]$:

$$|A_{T,k}(t)| \leq \frac{T^k}{k!} \sum_{i=0}^{r-k+l} \frac{(r+i+l)!}{i! (r+l)!}.$$

Expression (12) implies that, for any $\tilde{T} > 0$, there exists a constant C , such that, $\forall T \in [0, \tilde{T}]$, $\forall t \in [0, T]$

$$|\pi_{\Psi,T}(t)| \leq C \|\Psi\|. \quad (14)$$

Fig. 3 shows some of the Hermite polynomials $A_{T,k}$.

We use Hermite polynomials for defining the replanned trajectory y_p . To this end, define function

$$\begin{aligned} \bar{\psi} : \mathbb{R}^n \times \mathbb{R}^{m(r+l+1)} &\rightarrow \mathbb{R}^{m \times (r+l+1)} \\ (z, v, v_1, \dots, v_{r+l}) &\rightsquigarrow (\Phi_0, \Phi_1, \dots, \Phi_{r+l}), \end{aligned} \quad (15)$$

such that

$$\Phi_k = \psi_k(z, v_0, v_1, \dots, v_k), \quad k = 0, \dots, r+l, \quad (16)$$

where ψ_k is defined in (5). In this way, Φ_k represents the k -th derivative of the output obtained when the system state is z and the input and its derivatives are given by v_i , $i = 0, \dots, k$. Finally, define the reference trajectory

$$y_p(t) = \begin{cases} y_d(t) + \pi_{\Psi, T}(t) & \text{if } 0 \leq t < T \\ y_d(t) & \text{if } t \geq T, \end{cases}$$

where

$$\Psi = \bar{\psi}(x_0, u_0^{(0)}, \dots, u_0^{(r+l)}) - \bar{y}_d^{r+l}(0). \quad (17)$$

Function y_p solves the replanning problem, since: i) it belongs to C^{r+l} ; ii) it satisfies properties 1) and 2) because of conditions (10) and (17); iii) it satisfies property 3) because of condition (11).

Remark that, in this way, x_0 represents the initial state corresponding to output function y_p and $u_0, u_0^{(1)}, \dots, u_0^{(r+l)}$ represent the initial input and the initial input derivatives up to the degree $r+l$. The use of Hermite interpolation allows definition of replanned trajectories that corresponds to arbitrary conditions on the initial state, the initial input and its derivatives.

3.2 Iterative control law

Using the replanning method described in the previous paragraphs, the control law for system (7) is defined as follows:

$$u(t) = \phi_1(\bar{y}_p^{r+1}(t)) \quad (18)$$

where ϕ_1 is given in (3) and, for $t \in [kT, (k+1)T]$

$$y_p(t) = y_d(t) + \pi_{\Psi(k), T}(t - kT), \quad (19)$$

with

$$\Psi(0) = \bar{\psi}(x_0, u_0, u_0^{(1)}, \dots, u_0^{(r+l)}) - \bar{y}_d^{r+l}(0) \quad (20)$$

and, for $k > 0$,

$$\Psi(k) = \bar{\psi}(x(kT^-), \bar{u}^{r+l}(kT^-)) - \bar{y}_d^{r+l}(kT). \quad (21)$$

In (20), x_0 represents the initial state $x(0)$ and the assigned constants $u_0, u_0^{(1)}, \dots, u_0^{(r+l)}$ are the initial control input with its derivatives.

In other words, in time interval $[kT, (k+1)T]$ the control function u is used that would drive the nominal system (1) along the reference trajectory $y_p(t)$. This trajectory is computed by adding the polynomial function $\pi_{\Psi(k), T}$ to the reference trajectory y_d . In this way the replanned trajectory y_p satisfies the properties:

1. $\phi_0(\bar{y}_p^{r+1}(kT)) = x(kT)$, i.e. $x(kT)$ is the value at time kT of the state trajectory that corresponds to y_p ,

2. $\bar{y}_p^{r+l}((k+1)T^-) = \bar{y}_d^{r+l}((k+1)T)$, i.e. the replanned trajectory is the same as the desired trajectory at time $(k+1)T$.

3.3 Main results

A relevant property is that the resulting control function u is C^{l-1} continuous as shown in the following proposition.

Proposition 1. The control function u defined in (18) belongs to class C^{l-1} .

Proof. Since y_p is of class C^{r+l} in the open sets $[kT, (k+1)T]$ according to (4), the control function belongs to C^{l-1} in the union of intervals $[kT, (k+1)T]$, $k \in \mathbb{N}$. It remains to prove C^{l-1} continuity on kT , $k \in \mathbb{N}$. Since system (1) is flat, by definition (18) and taking (4) into account, it follows that, $\forall k \in \mathbb{N}$, $u^{(i)}(kT^+) = \phi_{i+1}(\bar{y}_p^{r+i+1}(kT^+))$, $i = 0, \dots, l-1$, moreover, by conditions (5), (21) $y_p^{(i)}(kT^+) = \psi_i(x(kT^-), \bar{u}^{r+i}(kT^-))$, $i = 0, \dots, l$, therefore by (6), $u^{(i)}(kT^+) = u^{(i)}(kT^-)$, $\forall i = 0, \dots, l-1$, $\forall k \in \mathbb{N}$, which proves C^{l-1} continuity. $\square$

Remark 1. With regards to Proposition 1, it is worth noting that integer l is, in practice, a free parameter provided that a sufficiently smooth desired trajectory $y_d \in C^{r+l}$ is designed. Consequently, this implies that the control input of the proposed method can be chosen as smooth as necessary or desired.

The main result of this paper requires the following Lipschitz assumption on function (1).

Assumption 1. Given flat system (1), there exist constants $0 < L_f, L_\psi \in \mathbb{R}$ for which $\forall x_1, x_2 \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ $\|f(x_1, u) - f(x_2, u)\| \leq L_f \|x_1 - x_2\|$, and the associated function $\bar{\psi}$ (see (15)) satisfies the following condition, $\forall x_1, x_2 \in \mathbb{R}^n$ and $u_0, \dots, u_{r+l} \in \mathbb{R}^m$

$$\|\bar{\psi}(x_1, u_0, \dots, u_{r+l}) - \bar{\psi}(x_2, u_0, \dots, u_{r+l})\| \leq L_\psi \|x_1 - x_2\|.$$

The following theorem states that it is always possible to choose a replanning time T , sufficiently small, such that the output tracking error is lower than any given positive constant ε .

Theorem 1. Let $\dot{x}(t) = f(x(t), u(t))$ be a control system with flat output (8), such that assumption 1 is satisfied. Let $y_d \in C^{r+l}(\mathbb{R}, \mathbb{R}^m)$ be a reference trajectory for the flat output ψ . Consider the differential system

$$\begin{cases} \dot{x}(t) = f(x(t), u(t)) + \eta(t) \\ x(0) = x_0, \end{cases} \quad (22)$$

where $\|\eta(t)\| \leq N$, $\forall t \in \mathbb{R}$, and the initial state x_0 is such that there exists an initial control u_0 for which $y_d(0) = \psi(x_0, u_0)$. Then, for any $\varepsilon > 0$, there exists $T > 0$ such that the solution of (22) with the control function u defined in (18) satisfies $\|\psi(x(t), u(t)) - y_d(t)\| \leq \varepsilon$, $\forall t \geq 0$.

Lemma 1. Let $L_f \in \mathbb{R}$ be such that $\forall x_1, x_2 \in \mathbb{R}^n$ and $\forall t \in \mathbb{R}$

Fig. 4. Top: The reference trajectory y_d , the replanned one y_p and the actual unicycle output y , for the unicycle example. Bottom: Simulation results for unicycle system with the iterative replanning method.

$$\|f(x_1, t) - f(x_2, t)\| \leq L_f \|x_1 - x_2\|, \quad (23)$$

and let x and x_r be the solutions of

$$\begin{cases} \dot{x}(t) = f(x, t) + \eta(t) \\ x(0) = x_0, \end{cases} \quad \text{and} \quad \begin{cases} \dot{x}_r(t) = f(x_r, t) \\ x_r(0) = x_0, \end{cases}$$

with $x_0 \in \mathbb{R}^n$ and $\|\eta(t)\| \leq N, \forall t \in \mathbb{R}$. Then

$$\|x(t) - x_r(t)\| \leq \frac{e^{L_f t} - 1}{L_f} N. \quad (24)$$

Proof. By hypothesis (23) the following differential inequality is satisfied

$$\begin{aligned} \frac{d\|x(t) - x_r(t)\|}{dt} &\leq \|f(x(t), t) + \eta(t) - f(x_r(t), t)\| \\ &\leq L_f \|x(t) - x_r(t)\| + N. \end{aligned}$$

Inequality (24) follows from the Comparison Lemma, solving the corresponding linear differential equation in the variable $\|x - x_r\|$. $\square$

Proof of theorem 1. For any $k \in \mathbb{N}$, $\forall t \in [kT, (k+1)T]$, let x_r be the solution of the following differential system:

$$\begin{cases} \dot{x}_r(t) = f(x_r(t), u(t)) \\ x_r(kT) = x(kT), \end{cases} \quad (25)$$

where the control u is given by (18). System $f(x, u)$ is flat and by definition 1 the associated function ϕ_0 satisfies the differential equation

$$\frac{d\phi_0(\bar{y}_p^r)}{dt}(t) = f(\phi_0(\bar{y}_p^r(t)), \phi_1(\bar{y}_p^{r+1}(t))). \quad (26)$$

By construction, the replanned trajectory y_p satisfies $x(kT) = \phi_0(\bar{y}_p^r(kT))$. Since u is defined by (18), differential equation (25) can be rewritten as

Fig. 5. Left: The control inputs for the iterative replanning method applied to the unicycle system. Right: The error functions for the iterative replanning method applied to the unicycle system.

$$\begin{cases} \dot{x}_r(t) = f(x_r(t), \phi_1(\bar{y}_p^{r+1}(t))) \\ x_r(kT) = x(kT). \end{cases}$$

Hence, for $t \in [kT, (k+1)T]$, $x_r(t)$ is the solution of the same differential equation (26) as ϕ_0 , therefore $x_r(t) = \phi_0(\bar{y}_p^r(t))$. Consequently, by (9), $\psi(x_r(t), u(t)) = \psi(\phi_0(\bar{y}_p^r), \phi_1(\bar{y}_p^{r+1})) = y_p(t)$. Moreover, using Lemma 1,

$$\begin{aligned} \|y(t) - y_p(t)\| &= \|\psi(x, u) - \psi(x_r, u)\| \\ &\leq L_\psi \|x - x_r\| \leq L_\psi \frac{e^{L_f T} - 1}{L_f} N. \end{aligned}$$

Set T_1 sufficiently small such that $L_\psi \frac{e^{L_f T_1} - 1}{L_f} N \leq \frac{\varepsilon}{2}$, so that

$$\|y(t) - y_p(t)\| \leq \frac{\varepsilon}{2}, \quad \forall t > 0. \text{ Remark that, } \forall k \in \mathbb{N},$$

$$\begin{aligned} &\|\bar{\psi}(x(kT), \bar{u}^{r+1}(kT)) - \bar{\psi}(x_r(kT), \bar{u}^{r+1}(kT))\| \\ &\leq L_\psi \|x(kT) - x_r(kT)\| \leq L_\psi \frac{e^{L_f T} - 1}{L_f} N. \end{aligned}$$

By (14) and Assumption 1, $\forall t \in [kT, (k+1)T]$

$$\begin{aligned} &\|y_p(t) - y_d(t)\| \\ &= \|\pi_{\bar{\psi}(x(kT), u(kT), \dots, u^{(r+1)}(kT)) - \bar{y}_d^{r+1}(kT), T}(t)\| \\ &\leq C \|\bar{\psi}(x(kT), u(kT), \dots, u^{(r+1)}(kT)) - \bar{y}_d^{r+1}(kT)\| \\ &= CL_\psi \|x(kT) - x_r(kT)\| \leq CL_\psi \frac{e^{L_f T} - 1}{L_f} N. \end{aligned}$$

Choose then T_2 such that $CL_\psi \frac{e^{L_f T_2} - 1}{L_f} N \leq \frac{\varepsilon}{2}$, so that

$\|y_p(t) - y_d(t)\| \leq \frac{\varepsilon}{2}$, $\forall t > 0$. Finally set $T = \min\{T_1, T_2\}$, then the thesis holds since, $\forall t > 0$ $\|y(t) - y_d(t)\| \leq \|y(t) - y_p(t)\| + \|y_p(t) - y_d(t)\| \leq \varepsilon$. $\square$

IV. SIMULATION RESULTS FOR A UNICYCLE SYSTEM

The unicycle kinematics are described by the following model:

$$\begin{cases} \begin{pmatrix} \dot{x} \\ \dot{z} \end{pmatrix} = v(t) \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} + \eta(t) \\ \dot{\theta} = \omega(t) + \eta_\theta(t), \end{cases} \quad (27)$$

where $(x, z) \in \mathbb{R}^2$ is the position of the center of the unicycle, θ is the orientation angle and v, ω are the velocity control inputs. Functions η and η_θ are noise terms that satisfy the bounds, $\|\eta(t)\| \leq N$, $|\eta_\theta(t)| \leq N_\theta$, $\forall t \in \mathbb{R}$. If the noise terms are not present, (i.e. $N = 0$ and $N_\theta = 0$) system (27) is called the nominal unicycle. In this example we want that the unicycle follows a C^2 curve with continuous control inputs $v(t), \omega(t) \in C^0$. We assume that the state is periodically measured with sample time $T = 1$ [s]. The performance output is given by $y = (x, z)$, which is a flat

Fig. 6. Simulation results for the unicycle with the Samson's method.

Fig. 7. Left: The control inputs for the Samson's method applied to the unicycle system. Right: The error functions for the Samson's method applied to the unicycle system.

Fig. 8. Schematic of a one-trailer system.

output with $r = 1$ for the nominal unicycle (27). In order to obtain a C^0 control input, by Proposition 1 we set $l = 1$. In this way the Hermite interpolating polynomials (13) have degree $2(r + l + 1) = 6$. As reference trajectory we consider a periodic spline y_d followed with constant speed 1 m/s. The noise bounds are given by $N = 0.5\sqrt{2}$ and $N_\theta = 0.5$. The obtained result is presented in Fig. 4. The control inputs and the error functions are depicted in Fig. 5, where the position error $e(t)$ and the angular error $e_\theta(t)$ are defined as follows $e(t) = \begin{pmatrix} x(t) \\ z(t) \end{pmatrix} - y_d(t)$, $e_\theta(t) = \theta(t) - \arg \dot{y}_d(t)$. Fig. 4 shows the reference trajectory y_d , the replanned one y_p and the actual unicycle output y .

4.1 Comparison with a well known method

In simulation, we have compared the method presented in Section III with the controller for the unicycle presented by Samson *et al.* in [20, p. 809]. The state appearing in the feedback control law presented in [20] is obtained through a discontinuous open loop observer which is updated at each observation time, set at $T = 1$ [s], as in the previous simulation. The noise bounds have the same values used in the previous simulation. The gains in this controller have been set to obtain control signals of magnitude similar to the method presented in this paper. The results obtained with the Samson's method are shown in Figs 6, 7.

The two methods show a similar performance in terms of tracking error. However, the control method presented in this paper has the advantage of providing input signals of an arbitrary degree of continuity, whereas the control signals of the classic controller are discontinuous (even if this is a consequence of having used a discontinuous observer). Moreover, the iterative output replanning method has the advantage of being applicable to any system with a flat performance output.

V. SIMULATION RESULTS FOR OF A ONE-TRAILER SYSTEM

The iterative output replanning method has been applied to a truck and trailer, or one trailer, system (see figure 8). With

Fig. 9. Top: Tracking results for the one-trailer system on a periodic spline, in the truck pulling trailer case. Bottom: The control inputs for the iterative replanning method applied to the one-trailer system, in the truck pulling trailer case.

respect to the coordinates (x, z) of the middle point of the trailer rear axle, the well-known equations for this system are given by:

$$\begin{cases} \dot{x} = v \cos \theta_1 \cos(\theta_0 - \theta_1) + \eta_x(t) \\ \dot{z} = v \sin \theta_1 \cos(\theta_0 - \theta_1) + \eta_z(t) \\ \dot{\theta}_0 = \omega + \eta_{\theta_0}(t), \\ \dot{\theta}_1 = \frac{v}{d} \sin(\theta_0 - \theta_1) + \eta_{\theta_1}(t) \end{cases}$$

where θ_0 and θ_1 are the orientation angles of the pulling truck and of the trailer respectively, and d represents the distance between the rear point (x, z) of the trailer and the joint point on the truck. The control input function v is the truck longitudinal velocity while ω represent its angular velocity. As shown in [21], this

Fig. 10. Left: The error functions for the iterative replanning method applied to the one-trailer system, in the truck pulling trailer case. Right: a close up of it on a smaller time interval.

system is flat ($r=2$) with respect to the performance output $y=(x, z)$. In this case, to keep small the degree of the interpolation polynomials, we have chosen $l=0$, obtaining therefore a discontinuous control by Proposition 1. In this simulation, the noise terms satisfy $|\eta_x(t)|$, $|\eta_z(t)|$, $|\eta_{\theta_0}(t)|$, $|\eta_{\theta_1}(t)| \leq 0.2$. The value of the distance d is set to be equal to 1 m.

5.1 Truck pulling a trailer

As reference trajectory y_d we consider the same C^2 periodic spline used for the unicycle example, characterized by a positive constant speed of 1 m/s. The obtained results are presented in figures 9 and 10, where $e_\theta(t) = \theta_1(t) - \arg \dot{y}_d(t)$. As in the case of the unicycle, the iterative output replanning method shows a good performance in terms of tracking error.

5.2 Truck pushing trailer

As is known, the trajectory tracking of trailer system in reverse gear is more difficult than in forward driving, because it is unstable (see [22]). Some examples of backward tracking for a one trailer system can be found in [23] and [24]. The reference trajectory is again the periodic spline used in the previous examples, with the difference, in this case, of being characterized by a constant negative speed of -1 m/s. The obtained results are presented in Figs 11 and 12.

VI. CONCLUSION

The paper has considered the output tracking problem for nonlinear system whose performance output is also a flat output of the system itself. The proposed method is based on an iterative output replanning that exploits the Hermite interpolating polynomials to achieve an overall arbitrarily smooth input and a tracking error that can be made arbitrarily small when the sampling period T is sufficiently reduced. The new technique appears especially useful for the motion control of autonomous vehicles

Fig. 11. Top: Tracking results for the one-trailer system on a periodic spline, in the truck pushing trailer case. Bottom: The control inputs for the iterative replanning method applied to the one-trailer system, in the truck pushing trailer case.

Fig. 12. Left: The error functions for the iterative replanning method applied to the one-trailer system, in the truck pushing trailer case. Right: a close up of it on a smaller time interval.

but other application fields may exploit the benefits of this hybrid feedforward/feedback approach.

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