Generalized bang–bang control for feedforward constrained regulation

Luca Consolini, Aurelio Piazzi *
Dipartimento di Ingegneria dell’Informazione, Università di Parma, Viale G.P. Usberti 181A, 43124 Parma, Italy

A R T I C L E   I N F O
Article history:
Received 7 August 2008
Received in revised form 28 February 2009
Accepted 29 June 2009
Available online 7 August 2009

Keywords:
Feedforward control
Generalized bang–bang control
Set-point constrained regulation
Input–output constraints
Minimum-time control
Linear programming
Linear systems

A B S T R A C T
In the behavioral framework for continuous-time linear scalar systems, simple sufficient conditions for
the solution of the minimum-time rest-to-rest feedforward constrained control problem are provided.
The investigation of the time-optimal input–output pair reveals that the input or the output saturates
on the assigned constraints at all times except for a set of zero measure. The resulting optimal input
is composed of sequences of bang–bang functions and linear combinations of the modes associated to
the zero dynamics. This signal behavior constitutes a generalized bang–bang control that can be fruitfully
exploited for feedforward constrained regulation. Using discretization, an arbitrarily good approximation
of the optimal generalized bang–bang control is found by solving a sequence of linear programming
problems. Numerical examples are included.

© 2009 Elsevier Ltd. All rights reserved.

1. Introduction

Recently in the control engineering literature, it has been em-
phasized that to achieve high performances in real applications,
due attention has to be paid to the constraints which all the plant
variables must comply with. In particular, the main approaches to
control system design with input and output constraints are the
following:

• Antiwindup and override feedback schemes. This is the stan-
dard approach in the practical industrial context; see, for exam-
ple, the recent book of Glattfelder and Schaufelberger (2003).

• Model predictive control. In the receding horizon strategy,
input constraints as well as output ones can be naturally con-
side red in designing the feedback controller; see, for instance,
Maciejowski (2002).

In this paper we address the subject of controlling a continuous-
time scalar linear system with input and output constraints by
setting a purely feedforward regulation problem to be solved in
minimum-time. We assume that the system is stable and want to
find a minimum-time feedforward input that brings the system
from a current rest condition to a new desired rest condition while
satisfying at all times given amplitude constraints on the input and
the output. In such a way, we can naturally deal with both actuator
limitations and overshooting and undershooting requirements.

It is well known that the minimum-time feedforward control
with input constraints only is given by the so-called bang–bang
control, i.e. the input signal switches between its extreme allowed
values (Lewis & Syrmos, 1995). In a behavioral setting, this paper
shows that in the presence of both input and output constraints
the minimum-time input–output pair enjoys the property that the
saturation of the input or the output signal occurs almost
everywhere. Therefore, the optimal feedforward input is given by
a sequence of bang–bang functions and linear combinations of the
system zero modes. This type of optimal control can be viewed
as a generalized bang–bang control. For the actual computation of
this time-optimal control, the proposed idea is to discretize the
continuous-time system and to solve the resulting discrete-time
problem by means of linear programming. In fact, in the discrete-
time case, input and output constraints can be represented as
linear inequalities and the minimum number of steps needed for
a rest-to-rest transition can be found with a sequence of linear
feasibility tests.

The idea of using linear programming for solving a minimum-
time problem for linear discrete-time systems subject to ampli-
tude input constraints dates back to Zadeh (1962). Subsequently,
various contributions have appeared by focusing on some im-
provements for this discrete-time problem (Bashein, 1971; Kim &
Engell, 1994; Scott, 1986). In this paper, we prove that the optimal

0005-1098/$ – see front matter © 2009 Elsevier Ltd. All rights reserved.
doi:10.1016/j.automatica.2009.06.030
discrete-time solution converges to the optimal continuous-time one when the sampling time approaches zero.

This article is structured as follows. In the second section the problem of minimum-time feedforward constrained regulation is presented for linear continuous-time systems. This is done in the framework of the behavioral approach (see Polderman and Willems (1998)). Herein the main result is a simple sufficient condition (Theorem 1) that guarantees the problem solvability. The third section is devoted to the study of the structure of the time-optimal solution. By exploiting the convexity of the system accessible set the main result (Theorem 2) is deduced. It states that the optimal input–output pair saturate on extreme values almost everywhere. From a corollary of this theorem (Corollary 1) the optimal feedforward input is then characterized as a generalized bang–bang control. In the fourth section the minimum-time constrained problem is introduced for discrete-time systems. A feasibility test is presented in Proposition 3 which is followed by an algorithm that computes the optimal discrete-time control through the solutions of a sequence of linear programming problems. Section 5 presents a convergence result. Theorem 4 shows that the optimal solution for the discretized system converges to the solution of the original continuous-time system as the sampling time goes to zero. Some simulation results are presented in Section 6. Conclusions are reported in Section 7.

Notation. $\mathcal{C}^1$ denotes the set of real functions defined over $\mathbb{R}$ that are continuous till the ith derivative. The ith order differential operator is $D^i$. The $L_\infty$ norm of a real function $f(t)$ defined and bounded over $\mathbb{R}$ is $\|f\|_{\infty} := \sup_{t \in \mathbb{R}} |f(t)|$ and the $L_1$ norm is $\|f\|_1 := \int_{-\infty}^{\infty} |f(t)|\,dt$. Given $x \in \mathbb{R}$, $|x| = \min\{z \in \mathbb{Z}: z \geq x\}$, $|x| = \max\{z \in \mathbb{Z}: z \leq x\}$. Given a subset $\mathcal{S} \subset \mathbb{R}^n$, $\mathcal{S}$ denotes the boundary of $\mathcal{S}$, cl($\mathcal{S}$) is the closure of $\mathcal{S}$. If $\mathcal{S} \subset \mathbb{R}^n$ is a Lebesgue measurable set then $|\mathcal{S}|$ denotes its measure. The space of locally integrable real functions is denoted by $L_1^\infty$.

2. The minimum-time feedforward constrained regulation problem

Consider a linear, stable, continuous-time system $\Sigma$ described by the scalar, strictly proper transfer function

$$H(s) = \frac{b(s)}{a(s)} = \frac{b_0s^n + b_{n-1}s^{n-1} + \cdots + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_0},$$

(1)

The system static gain is $H(0) = \frac{b_0}{a_0} \neq 0$ and the input and output are denoted by $u$ and $y$ respectively. Also, assume that polynomials $a(s)$ and $b(s)$ are coprime with $b_0 \neq 0$ and $n < m$. With $h(t)$ we denote the impulse response of system $\Sigma$, i.e. $h(t) = \mathcal{L}^{-1}[H(s)]$ where $\mathcal{L}^{-1}$ denotes the inverse Laplace transform. The behavior set of $\Sigma$ can be introduced as the set $\mathcal{B}$ of all input–output pairs $(u(\cdot), y(\cdot)) \in L_1^\infty \times L_1^\infty$ that are “weak” solutions of the differential equation (Polderman & Willems, 1998) ($a_0 := 1$):

$$\sum_{i=0}^{n} a_iD^iy = \sum_{i=0}^{m} b_id^iu.$$  

(2)

The control aim is to find a minimum-time feedforward input that causes a rest-to-rest transition from $y = 0$ to $y = y_f$ subject to arbitrarily assigned input and output constraints ($y_f \in \mathbb{R}$ is any desired output value). The rest condition of $\Sigma$ is characterized by the set of input–output equilibrium points designated as $\mathcal{E} := \{(u, y) \in \mathbb{R}^2: y = H(0)u\}$. We introduce, as a special subset of $\mathcal{B}$, the set $\mathcal{T}_P$ of all rest-to-rest transitions from $(0, 0) \in \mathcal{E}$ to $(y_f, y_f) \in \mathcal{E}$ subject to input and output constraints.

Definition 1. Let be given a constraint parameter set $\mathcal{P} := \{U_c, Y_c, y_f\}$ where $U_c = [u^c, u^c]$ and $Y_c = [y^c, y^c]$ are the constraint intervals for the input and output respectively and $y_f$ is the final output rest value for which

$$\left\{0, \frac{y_f}{H(0)} \right\} \subset U_c \quad \text{and} \quad \left\{0, y_f \right\} \subset Y_c. \quad (3)$$

Then define $\mathcal{T}_P$ as the set of all pairs $(u(\cdot), y(\cdot)) \in \mathcal{B}$ for which there exists $t_f > 0$ such that:

$$u(t) = 0 \quad \forall t < 0, \quad u(t) = \frac{y_f}{H(0)} \quad \forall t \geq t_f,$$

(4)

$$y(t) = 0 \quad \forall t < 0, \quad y(t) = y_f \quad \forall t \geq t_f,$$

(5)

$$y(t) \in Y_c \quad \forall t \in [0, t_f].$$

(6)

$$y(t) \in Y_c \quad \forall t \in [0, t_f].$$

(7)

The constraints intervals introduced in the above definition can encapsulate all the typical amplitude limitations that apply to the input and the output for any given set-point regulation problem. For example, if a regulation problem requires $|u(t)| \leq u_{\text{MAX}}$, $\forall t \in \mathbb{R}$, a maximum 10% overshooting and 5% undershooting on the output signal we can assign (consider $y_f > 0$): $U_c = [-u_{\text{MAX}}, +u_{\text{MAX}}]$, $Y_c = [-0.05y_f, +1.1y_f]$.

Lemma 1. Given system $\Sigma$ (1) and any $T > 0$, there exist two positive constants $M_u, M_y$ such that for any vector $w = [w_0, w_1, \ldots, w_{n-1}]^T \in \mathbb{R}^n$ and any $a \in \mathbb{R}$, there exists an input–output pair $(u(\cdot), y(\cdot)) \in \mathcal{B} \cap C^0$ such that:

1. $\|u\|_{\infty} \leq M_u \|w\|$, $\|y\|_{\infty} \leq M_y \|w\|$.

Proof. Set $g_i(t) = h^i(0)(T - t)^{i-1}(T - t)^{n-i}$, Functions $g_0(t)$, $g_1(t), \ldots, g_{n-1}(t)$ are linearly independent, therefore the following Gramian matrix is nonsingular

$$G = \int_0^T \begin{bmatrix} g_0(t) & g_1(t) & \cdots & g_{n-1}(t) \end{bmatrix} \frac{\partial}{\partial t} g_i(t)\,dt.$$  

Define the input

$$u(t) = \begin{cases} 0 & \text{if } t \in (-\infty, 0] \cup [T + \infty) \\ t^n(0)(T - t)^{n-1} g_0(t), g_1(t), \ldots, g_{n-1}(t) & \text{if } 0 \leq t \leq T. \end{cases}$$

This input signal belongs to $C^n$ and satisfies $D^iu(0) = D^iu(T) = 0$, $i = 0, \ldots, n - 1$. Define the output as follows

$$y(t) = \int_0^t h(t-\tau)u(\tau)\,d\tau \quad \text{if } t \geq 0.$$  

Clearly $y \in C^n$, and

$$y(T) = \int_0^T h(T - \tau)u(\tau)\,d\tau$$

$$= \int_0^T g_0(t)g_0(T), g_1(t), \ldots, g_{n-1}(t)\,d\tau G^{-1}w$$

$$= \{1, 0, \ldots, 0\}w = w_0.$$  

Moreover

$$D^iu(t) = \frac{\partial}{\partial t} \int_0^t h(t-\tau)u(\tau)\,d\tau \bigg|_{t=T} = \int_0^T Dh(t - \tau)u(\tau)\,d\tau + h(0)u(T),$$

and considering $u(T) = 0$, then

$$D^iu(T) = \int_0^T g_0(t)g_0(T), g_1(t), \ldots, g_{n-1}(t)\,d\tau G^{-1}w = \{0, 1, \ldots, 0\}w = w_1.$$  

Repeating the same procedure it follows that $D^iq(T) = w_i$, $i = 2, \ldots, n - 1$. Now define by time translation $u(t) = u(t-a)$ and $y(t) = y(t-a)$ so that $(\hat{u}, \hat{y}) \in \mathcal{B} \cap C^n$ and statements (1) and (2) of Lemma 1 are evidently verified. Finally, statement (3) holds since
\[\| \hat{u} \|_\infty = \| u \|_\infty \leq T^{de} (\| h \|_\infty + |Dh|) + \cdots + |D^{p-2} h| \| G^{-1} \| \| w \| ,\]

\[\| \hat{y} \|_\infty = \| y \|_\infty \leq \| h \| \| u \| ,\]

where \( \| h \|_1 = \int_0^\infty |h(t)| \, dt \) is the peak gain of system \( \Sigma \). \( \square \)

The following theorem gives a straightforward sufficient condition to ensure that \( T_p \) is not empty.

**Theorem 1.** Set \( T_p \) is not empty if

\[
\begin{bmatrix}
0, \frac{y_f}{H(0)} \\
H(0)
\end{bmatrix} \subseteq \left( u_i^-, u_i^+ \right) \quad \text{and} \quad \{0, y_f \} \subseteq \left( y_i^-, y_i^+ \right) .
\]

(8)

**Proof.** Without loss of generality we assume \( H(0) > 0 \) and \( y_f > 0 \). Let \( l(t) \) be any \( C^0 \) function such that

\[ l(t) = 0 \quad \forall t < 0, \quad l(t) = \frac{y_f}{H(0)} \quad \forall t > 1, \quad 0 \leq l(t) \leq \frac{y_f}{H(0)} \quad \forall t \in [0, 1].\]

Given a real constant \( \epsilon > 0 \), let the input to system \( \Sigma \) be given by \( l(\epsilon t) \) and denote by \( y(t; \epsilon) \) the corresponding output with \( y(t; \epsilon) = 0 \forall t < 0 \). Hence, the following limit holds:

\[ \lim_{\epsilon \to 0} \| y(t; \epsilon) - l(\epsilon t)H(0) \|_\infty = 0. \]

(9)

Indeed, the Laplace transform of \( y(t; \epsilon) - l(\epsilon t)H(0) \) is given by:

\[ L(s; \epsilon) \left( H(s) - H(0) \right), \]

(10)

where \( L(s; \epsilon) = L[l(\epsilon t)] \). Since \( H(s) - H(0) = s\tilde{H}(s) \), where \( \tilde{H}(s) \) is a suitable stable biproper transfer function, expression (10) can be written as \( L(s; \epsilon)\tilde{H}(s) = \tilde{H}(s)L \left[ \frac{1}{\epsilon}l(\epsilon t) \right] \). Therefore

\[ \| y(t; \epsilon) - l(\epsilon t)H(0) \|_\infty \leq \int_0^{\infty} |\tilde{H}(s)| \, ds \left\| \frac{d}{d\epsilon} l(\epsilon t) \right\| \| l(\epsilon t) \| \| H(0) \| \| u \| , \]

\[ \text{where } \tilde{h}(t) = \frac{d}{d\epsilon} l(\epsilon t) \text{ and } \int_0^{\infty} \tilde{h}(t) \, dt \text{ is the peak gain of } \tilde{H}(s). \]

Since \( \| d / d\epsilon l(\epsilon t) \|_\infty = \epsilon \| d / d\epsilon l(\epsilon t) \|_\infty \), limit (9) is proved.

Moreover, from \( \frac{d}{d\epsilon} l(\epsilon t) = \epsilon \cdot \frac{d}{d\epsilon} l(\epsilon t) \) we have

\[ \lim_{\epsilon \to 0} \left\| \frac{d}{d\epsilon} l(\epsilon t) \right\|_\infty = 0, \quad \epsilon = 1, \ldots, n - 1. \]

(11)

Then, using again the peak gain concept it follows that

\[ \lim_{\epsilon \to 0} \left\| \frac{d}{d\epsilon} y(t; \epsilon) \right\|_\infty = 0, \quad \epsilon = 1, \ldots, n - 1. \]

(12)

So far we have constructed a family of input–output pairs \( \{(l(t), y(t; \epsilon)) \} \in B \) parameterized by \( \epsilon > 0 \). Now, consider \( \epsilon < 1 \) and choose the “correcting pair” \( \{u(t; \epsilon), y(t; \epsilon)\} \in C^0 \cap B \) according to Lemma 1 with \( a = \epsilon^{-1} - 1, T = 1 \) and

\[ \hat{u}(t; \epsilon) = 0, \quad \text{for } t \in (\infty, 1, \epsilon^{-1} - 1] \cup [\epsilon^{-1}, +\infty), \]

\[ \hat{y}(t; \epsilon) = y_f - y(t; \epsilon), \]

\[ \frac{d}{d\epsilon} \hat{y}(t; \epsilon) \big|_{\epsilon=1} - \frac{d}{d\epsilon} y(t; \epsilon) \big|_{\epsilon=1} = 0, \quad \epsilon = 1, \ldots, n - 1. \]

Therefore the pair \( \{u(t; \epsilon), y(t; \epsilon)\} \in C^0 \cap B \) defined by

\[ \{u(t; \epsilon), y(t; \epsilon)\} = \{l(\epsilon t), y(\epsilon t); +u(t; \epsilon), y(t; \epsilon)\}, \]

satisfies the rest conditions at time \( t = \epsilon^{-1} \):

\[ \hat{u}(\epsilon^{-1}; \epsilon) = y_fH(0)^{-1}, \quad \frac{d}{d\epsilon} \hat{u}(t; \epsilon) \big|_{\epsilon=1} = 0, \quad \epsilon = 1, \ldots, m, \]

\[ \hat{y}(\epsilon^{-1}; \epsilon) = y_f, \quad \frac{d}{d\epsilon} \hat{y}(t; \epsilon) \big|_{\epsilon=1} = 0, \quad \epsilon = 1, \ldots, n - 1; \]

hence \( y(t; \epsilon) = y_f, \forall t \geq \epsilon^{-1} \). Because of (9) and (12) and statement (3) of Lemma 1, it follows that

\[ \lim_{\epsilon \to 0} \| y(t; \epsilon) \|_\infty = 0, \]

\[ \lim_{\epsilon \to 0} \| y(t; \epsilon) \|_\infty = 0. \]

(14)

(15)

By virtue of (8), \( |u_i^-| > 0 \) and \( u_i^- - \frac{y}{H(0)} > 0 \) so that there exists \( \epsilon_u > 0 \) such that, by (14), \( \forall \epsilon < \epsilon_u \)

\[ \| y(t; \epsilon) \|_\infty \leq \min \left\{ |u_i^-|, u_i^- - \frac{y}{H(0)} \right\}. \]

(16)

By virtue of (8), \( |y_i^-| > 0 \) and \( y_i^- > y_f \) and by (9) there exists \( \epsilon_y > 0 \) such that \( \forall \epsilon < \epsilon_y \)

\[ \| y(t; \epsilon) \|_\infty \leq \frac{\min \{ |y_i^-|, y_i^- - y_f \} }{2}. \]

(17)

and by (15) there exists \( \epsilon_2 > 0 \) such that \( \forall \epsilon < \epsilon_2 \)

\[ \| y(t; \epsilon) \|_\infty \leq \frac{\min \{ |y_i^-|, y_i^- - y_f \} }{2}. \]

Finally setting \( \epsilon_0 = \min \{ \epsilon_u, \epsilon_y, \epsilon_2 \} \) we obtain that

\[ (u(t; \epsilon_0), y(t; \epsilon_0)) \in T_p. \]

(18)

**Remark 1.** Note that sufficient condition (8) differs from assumption (3) of Definition 1 defining \( T_p \) just for the exclusion of the four endpoints of intervals \( U_i \) and \( Y_i \). Hence, condition (8) implies that there exists at least a small distance between the constraints extrema and the corresponding steady-state input–output values. This permits to construct (as shown in the proof) an input–output pair that reaches the steady-state in finite time while respecting the constraints.

Once inclusions (8) are satisfied, the emerging natural problem is to determine among all the constrained transitions of \( T_p \) the fastest one, i.e. the optimal rest-to-rest transition with associated minimum transition time \( t_f^* \):

\[ t_f^* := \inf_{(u_i, y_i) \in T_p} T_f(u_i, y_i) \]

(19)

where \( T_f \) is the following functional

\[ T_f(u_i, y_i) = \inf \left\{ t_1 : u(t) = \frac{y_i}{H(0)} \right\}, \quad y(t) = y_f, \forall t \geq t_1 \]

(20)

which is well defined by Definition 1. Note that \( t_f^* \) corresponds to the minimum \( T_f(u_i, y_i) \) that is achievable with an optimal pair \( (u_i^*, y_i^*) \) that is essentially unique in \( T_p \) (proofs are reported in Appendix A).

On the other hand, from a control viewpoint the problem is to directly determine the optimal feedforward input \( u_i^*(t) \) that corresponds to minimum-time \( t_f^* \). An approximate solution to this problem using linear programming is exposed in Section 5.

**3. Characterization of the time-optimal solution for the continuous-time case**

This section gives a characterization of the time-optimal solution \( (u_i^*(t), y_i^*(t)) \in T_p \) to the constrained set-point regulation problem for continuous-time systems.

**Definition 2.** Consider a linear system of the form

\[ \dot{x} = Ax + bu, \quad y = cx, \]

where \( x \in \mathbb{R}^n, u \in \mathbb{R}, y \in \mathbb{R} \) and intervals \( U_i, Y_i \) are given constraints on the input and the output respectively. Then, the constrained reachable set at final time \( T \), starting from initial state \( x_0 \) is denoted by \( A_{U, Y}(x_0, T) \) and is defined as
Theorem 2. There exist open, nonempty, nonoverlapping intervals \( I_i, O_j \subset \mathbb{R}, i, j \in \mathbb{N} \) and real coefficients \( a_{0i}, a_{1i}, \ldots, a_{ni}, \beta_{0j}, \beta_{1j}, \ldots, \beta_{nj} \) such that
\[ 1.0, t_0' \} \cup \bigcup_{j} \text{c}(O_j); \]
\[ u^*(t) = u_i^- \quad \forall t \in I_i \quad \text{or} \quad u^*(t) = u_i^+ \quad \forall t \in I_i, \]
\[ y^*(t) = \alpha_{0i} + \sum_{k=0}^{n} \alpha_{ki}m_k^i(t) \quad \forall t \in I_i, \]
\[ y^*(t) = \beta_{0j} + \sum_{i=1}^{m} \beta_{ij}m_i^j(t) \quad \forall t \in O_j, \]

\[ \delta = \left\{ t \in [0, t_0'] : u^*(t) \not\in \{u_i^-, u_i^+\}, \text{ and } y^*(t) \not\in \{y_i^-, y_i^+\} \right\} \]

\[ \delta = \bigcup_{i} I_i, \quad \sum_{i} |I_i| \neq 0. \]

In particular there exists an integer \( l \) such that \( I_l = [a, b], \) with \( b - a > 0 \) and \( \forall t \in I_l, u^*(t) \in (u_i^-, u_i^+), y^*(t) \in (y_i^-, y_i^+). \) Thus there exists \( \delta > 0 \) such that \( u^*(t) \in (u_i^-, u_i^+ - \delta), y^*(t) \in (y_i^-, y_i^+ - \delta), \forall t \in I_l. \) By the principle of optimality, state \( x_0 := x(a) \) belongs to the boundary of the constrained reachable set from \( x_0 := x(a) \) after a time \( b - a, \) that is \( x_0 \in \partial \mathbb{A}_{U_i,Y_i}(x_0, b - a). \)

By Proposition 1, \( \mathbb{A}_{U_i,Y_i}(x_0, b - a) \) is a convex set, therefore the supporting hyperplane at \( x_0, \) defined by \( \{x \in \mathbb{R}^n : p + q^T x = 0 \} \) with
\[ p + q^T x_0 = 0, \]
p satisfies the inequality
\[ p + q^T x \leq 0 \quad \forall x \in \mathbb{A}_{U_i,Y_i}(x_0, b - a). \]

Hence, for any function \( u(t) : [a, b] \rightarrow \mathbb{R} \) such that \( u(t) \in U_i \) and \( y(t) \in Y_i, \forall t \in [a, b] \) it follows that
\[ p + q^T \left\{ e^{A(b-a)}x_0 + \int_{a}^{b} e^{A(t-s)} u(t) \, dt \right\} \leq 0 \]
where the equality holds for the optimal control \( u^*(t). \) Choose any \( x_0 \in \mathbb{R}^n \) satisfying
\[ p + q^T x_0 > 0. \]
4. The minimum-time problem for discrete-time systems

In this section the minimum-time feedforward control problem is restated for discrete-time systems and a solution is provided using linear programming. Consider a linear discrete-time system \( \Sigma_d \) described by the scalar strictly proper transfer function

\[
H_d(z) = \frac{b(z)}{a(z)} = \frac{b_m z^m + b_{m-1} z^{m-1} + \cdots + b_0}{z^n + a_{n-1} z^{n-1} + \cdots + a_0}.
\]  

(27)

Assume that \( \Sigma_d \) is stable, \( a(z) \) and \( b(z) \) are coprime and \( H_d(1) \neq 0 \). The input and output sequences are denoted by \( u(k) \) and \( y(k) \) respectively, \( k \in \mathbb{Z} \). The behavior \( B_d \) of system \( \Sigma_d \) is the set of all input–output pairs \((u(\cdot), y(\cdot))\) satisfying the associated difference equation:

\[
y(k + n) + a_{n-1} y(k + n - 1) + \cdots + a_0 y(k) = b_m u(k + m) + b_{m-1} u(k + m - 1) + \cdots + b_0 u(k). \tag{28}
\]

The set of input–output equilibrium points of \( \Sigma_d \) is \( \mathcal{E}_d = \{(u, y) \in \mathbb{R}^2 : y = H_d(1) u\} \) and the set \( \mathcal{K}_p \subseteq B_d \) of all rest-to-rest constrained transitions from \((0, 0) \in \mathcal{E}_d \) to \((y_f^1, y_f^2) \in \mathcal{E}_d \) is defined as follows.

**Definition 3.** Let be given a constraint parameter set \( p := \{u_c, y_c, y_f\} \) where \( u_c = [u_c^1, u_c^2] \) and \( y_c = [y_c^1, y_c^2] \) are the constraint intervals for the input and output respectively and \( y_f \) is the final output rest value for which

\[
[0, y_f] \subset Y_c \quad \text{and} \quad \left\{ 0, \frac{y_f}{H_d(1)} \right\} \subset U_c.
\]

Then define \( \mathcal{K}_p \) as the set of all pairs \((u(\cdot), y(\cdot))\) in \( B_d \) for which there exists \( k_f \in \mathbb{N} \) such that:

\[
\begin{aligned}
&u(k) = 0 \quad \forall k < 0, \quad u(k) = \frac{y_f}{H_d(1)} \quad \forall k \geq k_f, \\
&u(k) \in U_c \quad k = 0, \ldots, k_f - 1, \\
&y(k) = 0 \quad \forall k < 0, \quad y(k) = y_f \quad \forall k \geq k_f, \\
&y(k) \in Y_c \quad k = 0, \ldots, k_f - 1.
\end{aligned}
\]  

(30)

The resulting form is the discrete counterpart of Theorem 1. Its proof is analogous to that of Theorem 1 and is omitted for brevity.

**Theorem 3.** Set \( \mathcal{K}_p \) is not empty if

\[
\left\{ 0, \frac{y_f}{H_d(1)} \right\} \subset (u_c^1, u_c^2) \quad \text{and} \quad [0, y_f] \subset (y_c^1, y_c^2). \tag{33}
\]

The minimum-time feedforward constrained control problem for discrete-time systems consists in finding the optimal input sequence \( u^*(k), k = 0, 1, \ldots, k_f^* - 1 \) for which the pair \((u^*(\cdot), y^*(\cdot)) \in \mathcal{K}_p \) is a minimizer for the optimization problem:

\[
k_f^* = \min_{(u(\cdot), y(\cdot)) \in \mathcal{K}_p} K_f(u(\cdot), y(\cdot)). \tag{34}
\]

**Proposition 3.** The set \( \mathcal{K}_p \) of all rest-to-rest constrained transitions is not empty if and only if there exist \( k_f \in \mathbb{N} \) and a vector \( u \in \mathbb{R}^n \) for which the following linear programming (LP) problem is feasible:

\[
\begin{align*}
&u^*_c \cdot 1_n \leq u \leq u^*_f \cdot 1_f, \tag{35} \\
y^*_c \cdot 1_f \leq y \leq y^*_c \cdot 1_f, \tag{36} \\
&\mathbf{H} \left[ \begin{array}{c} y_f^1 \\ y_f^2 \\ \vdots \\ y_f^{1_n} \\ y(k_f + 1) \\ \vdots \\ y(k_f + n - 1) \\ u \\ H_d(1) \cdot 1_n \end{array} \right] = \mathbf{y}.
\end{align*}
\]

(37)

where \( \mathbf{H} \in \mathbb{R}^{k_f \times k_f + n} \) is defined by \( H_f := h_d(i - j) \) and \( \mathbf{H} \in \mathbb{R}^{k_f \times k_f + n} \) by \( H_f := h_d(i + k_f - j) \).

**Proof (Sufficiency).** Assume that there exist \( k_f \in \mathbb{N} \) and a vector \( u = [u_0, u_1, \ldots, u_{k_f - 1}]^T \) for which Eqs. (35)-(37) are satisfied. Define the input sequence

\[
u(k) = \begin{cases} 0 & \text{if } k < 0 \\ u_k & \text{if } 0 \leq k < k_f \\ \frac{y_f}{H_d(1)} & \text{if } k \geq k_f,
\end{cases}
\]

(38)

which satisfies Properties (29) and (30) of Definition 3. The output is given by \( y(k) = \sum_{i=0}^{\infty} u(k - i) h_d(i) \). Setting \( y = [y_0, y_1, \ldots, y_{k_f - 1}]^T \in \mathbb{R}^{k_f} \) and \( y = [y_0, y_1, \ldots, y_{k_f - 1}]^T \in \mathbb{R}^{k_f} \),

according to \( y = Hu \) \( \tilde{y} = H \left[ \begin{array}{c} y_f^1 \\ y_f^2 \\ \vdots \\ y_f^{1_n} \\ y(k_f + 1) \\ \vdots \\ y(k_f + n - 1) \end{array} \right] \) it follows that

\[
y(i) = y_r, \quad i = 0, 1, \ldots, k_f - 1
\]

(39)

and, by (36), sequence \( y(k) \) satisfies the constraint (32) of Definition 3. It remains to show that \( y(i) = y_r, \forall i \geq k_f + n \). To prove this, set \( k = k_f \) in difference equation (28). By iteration we have \( y(k) = y_r, \forall k > k_f + n \). Indeed condition (37) guarantees that at \( k = k_f \) the system has reached the equilibrium.

(Necessity). Assume that the set \( \mathcal{K}_p \) is nonempty, therefore there exists \( k_f \in \mathbb{N} \) and a pair \((u(k), y(k))\) which satisfies conditions (29)-(32). Define \( u = [u(0), u(1), \ldots, u(k_f - 1)]^T \), then (35) follows from (30) and inequality (36) follows from (32) and the fact that \( y(0), y(1), \ldots, y(k_f - 1) \) can be found. Finally (37) follows from (31) and the fact that

\[
\begin{pmatrix}
y(k_f) \\
y(k_f + 1) \\
\vdots \\
y(k_f + n - 1) \\
u \\
H_d(1) \cdot 1_n
\end{pmatrix}
= \mathbf{H}
\]

By virtue of Proposition 3, the minimum number of steps \( k_f^* \) and an associated optimal feedforward input \( u^*(k), k = 0, 1, \ldots, k_f^* - 1 \) can be determined by means of a sequence of LP feasibility tests (the problem defined at (35)-(37)) through the simple bisection algorithm reported below. In this algorithm \( \text{LPP} (p, k_f, u) \) denotes a linear programming procedure that solves problem (35)-(37): if the problem is feasible it returns a Boolean true value along with a solution \( u \in \mathbb{R}^n \). This solution vector \( u \) defines a corresponding input sequence according to

\[
[u(0), u(1), \ldots, u(k_f - 1)]^T = u.
\]

(39)

**Minimum-time feedforward constrained regulation algorithm**

**Input:** \( H_d(z) \) and \( p = \{U_c, Y_c, Y_f\} \)

**Output:** \( k_f^* \) and \( u^* \) that corresponds to an optimal control sequence \( u^*(k) \) according to (39).
\[ k_f \leftarrow 1 \]
\[ l \leftarrow 0 \]

\textbf{while} \sim \text{LPP}(p, k_f, u) \text{ do}

\[ l \leftarrow k_f \]
\[ k_f \leftarrow 2k_f \]

\textbf{end while}

\[ h \leftarrow k_f \]

\textbf{while} \ h - l > 1 \text{ do}

\[ k_f \leftarrow \frac{h + l}{2} \]

\textbf{if} \ \sim \text{LPP}(p, k_f, u) \text{ then} \[ l \leftarrow k_f \]

\textbf{else} \[ h \leftarrow k_f \]

\textbf{end if}

\textbf{end while}

\[ k_f^* \leftarrow h \]
\[ u^* \leftarrow u \]

\textbf{Remark 2.} Differently from the continuous-time case, the discrete-time optimal solution \( u^*(k) \) is not unique (see Desoer and Wing (1961)).

5. An approximated solution to the continuous-time problem using discretization

The procedure developed in Section 4 allows to find the optimal minimum-time constrained transition for discrete-time systems. This section shows that it can be used to find an approximate solution to the continuous-time problem. Given the continuous-time system \( \Sigma \) (1), an approximation to the optimal bang-bang control \( u^*(t) \) can be found as follows:

- Choose a sampling period \( T \) and determine the discretized system using a zero-order equivalence, by relation \( H_T(z) = (1 - z^{-1})Z \left[ \frac{H(s)}{s} \right] \), where \( Z[P(s)] = \sum_{i=0}^{+\infty} p(iT)z^{-i} \) and \( p(t) = \mathcal{L}^{-1}[P(s)] \) is the impulse response of a system with transfer function \( P(s) \).
- Find a minimum-time input sequence \( u^*_T(k) \), using the algorithm described in Section 4.
- An approximated continuous-time solution is given by

\[ u^*_T \left( \left[ \frac{t}{T} \right] \right). \tag{40} \]

The following result shows that solution (40) can be made arbitrarily close to the optimal one, by choosing a sufficiently small sampling period \( T \).

\textbf{Theorem 4.} Assume that inclusions (8) of Theorem 1 are satisfied. Let \( t^*_f \) be the optimal time as defined in (16) and let \( (u^*, y^*) \) be the associated optimal pair. Let \( k^*_T(T) \) be the minimum number of steps defined by (34) relative to system \( H_T(z) \) and let \( (u^*_T, y^*_T) \) be the associated optimal sequence pair. Then the following limits hold

\[ \lim_{T \to 0} k^*_T(T)T = t^*_f, \tag{41} \]
\[ \lim_{T \to 0} u^*_T \left( \left[ \frac{t}{T} \right] \right) = u^*(t), \quad \text{a.e.} \tag{42} \]
\[ \lim_{T \to 0} y^*_T \left( \left[ \frac{t}{T} \right] \right) = y^*(t), \quad \text{a.e.} \tag{43} \]

\textbf{Proof.} Limit (41) is equivalent to the following two inequalities

\[ \lim_{T \to 0} k^*_T(T)T \geq t^*_f \tag{43} \]
\[ \lim_{T \to 0} k^*_T(T)T \leq t^*_f. \tag{44} \]

First, to prove (43), we assume by contradiction that there exists \( \sigma > 0 \) for which

\[ t^*_f - \liminf_{T \to 0} k^*_T(T)T = \sigma \tag{45} \]

and show that, as a consequence, there exists a continuous-time input–output pair that performs the constrained rest-to-rest transition in a time less than \( t^*_f \). By assumption (45) there exists an infinite sequence of decreasing sampling times \( T_i > 0, i \in \mathbb{N} \), such that \( \lim_{i \to \infty} T_i = 0 \) and the following two properties are verified

\[ \lim_{i \to \infty} t^*_f - k^*_T(T_i)T_i = \sigma, \tag{46} \]

\[ t^*_f - k^*_T(T_i)T_i \in \left[ \frac{3}{\sigma}, \frac{5}{4} \right], \quad \forall i \in \mathbb{N}. \tag{47} \]

Set \( c = \left( \| h(0^+) \| + \| Dh(0) \| \right) \max\{\| u^*_i \|, \| u^*_i \| \} \), \( y_m = \min \{ \| y^*_i \|, \| y^*_i \| \} \) and define the continuous-time input \( u_i(t) = u^*_T(\{ t/T_i \}) \frac{y_m - y_i}{y_m} \). If the corresponding output is given by \( y_i(t) = \int_0^t h(t - \nu) u_i(\nu) \, d\nu \) then it satisfies the property \( y_i(kT_i) = y^*_T(k) \frac{y_m - y_i}{y_m} \in [y^*_T, y^*_T] \), \( \forall k \in \mathbb{Z} \). By Lemma 4 (see Appendix B), \( \forall t \geq 0, \forall i \in \mathbb{N} \):

\[ y_i(t) \leq y^*_T + cT_i \frac{y_m - y^*_i}{y_m}, \leq y^*_T, \quad \text{a.e.} \tag{48} \]

and analogously \( y_i(t) \geq y^*_T \). Therefore the pair \( (u_i, y_i) \) satisfies the input–output constraints and reaches final rest conditions because \( \forall t \geq T_i k^*_T(T_i), u_i(t) = y^*_T(\{ t/T_i \}) \frac{y_m - y_i}{y_m} \). But, \( i \notin \mathbb{T} \) so that to enforce the required final rest conditions, in time interval \( [T_i k^*_T(T_i), T_i k^*_T(T_i) + \sigma/2] \) we add a correcting term to the input \( u_i(t) \) as follows. Apply Lemma 5 (Appendix B) to find a correcting pair \( (\hat{u}_i, \hat{y}_i) \) such that

\[ \hat{u}_i(t) = 0 \quad \text{if} \quad t < T_i k^*_T(T_i), \hat{u}_i(t) = y^*_T(\{ t/T_i \}) \frac{y_m - y_i}{y_m}, \quad \text{if} \quad t > T_i k^*_T(T_i) + \sigma/2 \]

\[ D\hat{y}_i(T_i k^*_T(T_i) + \sigma/2) = 0 \]

Then define the pair \( (\tilde{u}_i, \tilde{y}_i) = (u_i + \hat{u}_i, y_i + \hat{y}_i) \) for which \( \tilde{y}_i(t) = y_i, \forall t \geq T_i k^*_T(T_i) + \sigma/2 \). Moreover in the interval \( T_i k^*_T(T_i) < t < T_i k^*_T(T_i) + \sigma/2 \)

\[ \hat{u}_i(t) \leq y_i H(0)^{-1} \left( \frac{y_m - cT_i}{y_m} + M_{\Delta} \frac{cT_i}{y_m} \right), \]

\[ \tilde{y}_i(t) \leq y_i - cT_i \frac{y_m - y^*_i}{y_m} + M_{\Delta} \frac{cT_i}{y_m}, \quad \text{where} \quad M_{\Delta} = \text{const}, \text{not the length of the correction is given by} \frac{\sigma}{4} \text{and is fixed for all } i \text{. By choosing a sufficiently small } T_i, \text{ the input and output constraints can always be satisfied. Therefore, there exists a continuous-time input–output pair that performs the constrained rest-to-rest transition in time } T_i k^*_T(T_i) + \frac{\sigma}{2}. \text{ Hence, by (47), } T_i k^*_T(T_i) + \frac{\sigma}{2} \leq t^*_f - \frac{\sigma}{4} < t^*_f. \text{ This last inequality contradicts the optimality of } t^*_f \text{ so that proof of (43) is completed. In order to prove limit (44), assume by contradiction that there exists } \sigma > 0 \text{ for which}

\[ \limsup_{T \to 0} k^*_T(T)T - t^*_f = \sigma. \tag{48} \]

Hence, there exists an infinite sequence of decreasing sampling times \( T_i > 0, i \in \mathbb{N} \), such that \( \lim_{i \to \infty} T_i = 0 \) and

\[ k^*_T(T_i)T_i - t^*_f \in \left[ \frac{3}{\sigma}, \frac{5}{4} \right], \quad \forall i \in \mathbb{N}. \tag{49} \]
Finally, define the corrected input–output pair by \( \tilde{u}_t, \tilde{y}_t \) = \( (\tilde{u}_t + \bar{u}_t, \tilde{y}_t + \bar{y}_t) \). Then, \( \tilde{y}_t(k) = \tilde{y}_t(k) \) if \( k < k_t \), otherwise \( \tilde{y}_t(k) = y_t(k) \) for \( k \geq k_t \). It remains to show that the input and output constraints are satisfied for \( k_t \leq k < k_t + nl \). Consider that \( \forall k \geq k_t, |y_t(k) - y_t(k)| \leq |y_t(k) - \bar{y}_t(k)| + \|\tilde{r}_k\| \leq \epsilon + \frac{\epsilon M}{H(0)} \). Therefore, for \( k_t \leq k < k_t + nl \),

\[
y_t(k) - y_t(k) \leq \frac{\epsilon M}{H(0)} \leq \frac{\epsilon M}{H(0)} y_m,
\]

\[
y_t(k) - y_t(k) \leq \frac{\epsilon M}{H(0)} y_m + B_e.
\]

Term \( \|W(T, l)^{-1}\| \) is bounded for any \( T_r > 0 \) because quantity \( T_r l \) is included in a compact interval according to \( T_r / l \in \left( \frac{T_r}{T_r - T_r^*}, \frac{T_r}{T_r + T_r^*} \right) \) and \( W(T, l) \) is a continuous function of its argument. This means that \( \lim_{T_r \to 0} \|W(T_r, l)\| = 0 \). Choose \( \epsilon > 0 \) (and consequently \( T_r \)) sufficiently small such that \( (\hat{u}_t, \hat{y}_t) \) satisfies the input and output constraints and \( |\hat{r}_T/T_r| < \sigma/4 \), i.e., \( k_r T_r - t^*_r < \sigma/4 \). Considering the introduced sequence \( [T_r] \) of decreasing sampling times, there exists \( r \in \mathbb{N} \) such that \( T_r < T_r^* \) and

\[
k_r^r T_r - t^*_r < \sigma/4.
\]

Pair \( (\hat{u}_t, \hat{y}_t) \) satisfies the input and output constraints and performs the required rest-to-rest transition in \( k_r + nl \) steps. Taking into account that \( nl \leq \sigma/4r \), from (51) \( (k_r + nl) T_r - t^*_r < \sigma/2 \) and from (52) \( k_r^r T_r - t^*_r \geq (3/4)\sigma \). Therefore, \( k_r^r T_r > k_r + nl \) and this violates the optimality of \( k_r^r T_r \). This completes the proof of (44) and therefore (41) holds.

Let \( T_r \) be a sequence of decreasing sampling times such that \( \lim_{n \to \infty} T_r = 0 \). Hence, limit (41) holds, i.e., \( \lim_{n \to \infty} k_r^r T_r = T_r^* \). Consider the pairs \( (u_t^n, y_t^n) \) and apply the procedure devised in the first part of this proof to obtain continuous-time pairs \( (\hat{u}_t, \hat{y}_t) \) for which, when \( i \to \infty \), the transition time \( T^*_i \) converges to \( T^*_r \). By Proposition 8 in Appendix A, the sequence \( (\hat{u}_t, \hat{y}_t) \) converges a.e. to the unique optimal pair \( (u_t^*, y_t^*) \) as \( i \to \infty \). Since \( \lim_{n \to \infty} \|u_t^n\|_\infty \to \|u_t^*\|_\infty \) and \( \lim_{n \to \infty} \|y_t^n\|_\infty \to \|y_t^*\|_\infty \), the pairs \( (u_t^n, y_t^n) \) converge a.e. to \( (u_t^*, y_t^*) \) when \( i \to \infty \) and therefore (42) holds. \( \square \)

6. Examples

**Example 1.** Consider a continuous-time system described by transfer function \( H(s) = \frac{10(s + 2)}{s(s + 1)(s + 2) + 10(s + 2)} \). We desire a rest-to-rest transition from \( y = 0 \) to \( y = 3(\pi y) \) to be completed in minimum-time with amplitude input constraints defined by \( U = [u^L, u^U] = [-1.8, 1.8] \). In a first case no output constraints are considered, i.e., \( Y = (-\infty, \infty) \), and in a second case we impose \( Y = [-0.1, 3.1] \). This corresponds to regulation constraints given by a maximum 3.3% overshooting and 3.3% undershooting. The system static gain is \( H(0) = 2 \) and conditions (8) of Theorem 1 are satisfied: \( 0.15 < (-1.8, 1.8), 0.3 < (-\infty, \infty) \) and \( 0.15 < (-1.8, 1.8), 0.3 < (-\infty, 3.1) \). Hence the minimum-time feedforward constrained regulation problem has solution in both cases. The optimal control \( u_t^*(t) \) is computed by applying the discretization procedure of Section 5 with sampling period \( T = 0.002 \) s.
Consolini, Given any sequence of input–output pairs $P$ and $P_{old}$, $P$ is satisfied. The optimal input–output pair is due to the intrinsic difficulty in regulating a system with both an unstable zero and a couple of purely imaginary zeros.

Fig. 1. Example 1, bang-bang control.

Fig. 2. Example 1, generalized bang-bang control.

The results are exposed in Figs. 1 and 2. Both figures plot the pair $(u^*(\cdot), y^*(\cdot))$ over the optimal transition interval. Fig. 1 shows that $u^*(\cdot)$ is the well-known bang–bang control that permits to obtain the minimum-time $t^*_p = 0.6805$ s at the price of a large overshoot (more than 100% of the final rest value). In the second case, due to the imposed output constraints, the overshooting is almost completely removed (see Fig. 2) and the resulting optimal feedforward control $u^*(\cdot)$ is composed of a bang–bang function followed by a zero dynamics mode (in which the output saturates the constraint) and a final short bang–bang spike. The associated minimum-time is $t^*_p = 1.898$ s.

Example 2. This last example considers a system with transfer function $H(s) = \frac{10(3.5+\pi)(s^2+25)}{(s^2+2)(s^2+3)(s+4)(s+5)}$. The required rest-to-rest transition is from $y = 0$ to $y = 3$. The constraint intervals are $U_e = [-2, +2]$ and $Y_e = [-0.1, +3.1]$. Again conditions (8) of Theorem 1 are satisfied. The optimal $u^*(\cdot)$ and the corresponding $y^*(\cdot)$ are plotted in Fig. 3. The input is composed of a bang–bang spike, a zero mode function and a bang–bang function. The achieved minimum-time is $t^*_p = 1.382$ s. The sampling time used for the computation is $T = 0.002$ s. It is worth noting the intricate behavior of the optimal $y^*(\cdot)$: after a relatively long time plateau the output increases till to a local maximum, then decreases till to a local minimum and finally reaches the desired rest position. The surprising details of the optimal input–output pair are due to the intrinsic difficulty in regulating a system with both an unstable zero and a couple of purely imaginary zeros.

Fig. 3. Example 2, generalized bang-bang control.

7. Conclusions

This paper has posed a new minimum-time feedforward regulation problem with input and output amplitude constraints. The provided solution leads to a generalization of the classic bang–bang control that can be determined by means of a discretization procedure based on linear programming feasibility tests. A novelty of the proposed approach to constrained regulation is the ability to deal with both (i) arbitrarily stringent constraints on input and output and (ii) nonminimum-phase plants with purely imaginary zeros. This implies a significant improvement over the inversion-based approach to feedforward constrained regulation (Piazzi & Visioli, 2001; 2005).

An interesting extension of the proposed approach would be the MIMO (multi-input multi-output) case. Conceptually, the MIMO solution should still exhibit a generalized bang–bang structure (i.e. almost at all times at least one of the inputs or one of the outputs saturates the constraint). However possible degeneracies may emerge in the non-square case (when the number of inputs and outputs are different). This will be investigated in future research. The generalized bang–bang control seems a technique that can be applied to a broad range of applications. First results in process control and mechatronics have recently appeared (Consolini, Gerelli, Guarino Lo Bianco, & Piazzi, 2009; Consolini, Piazzi, & Visioli, 2007).

Appendix A. Existence and uniqueness of the solution to the minimum-time feedforward constrained regulation problem

First we recall a result from Polderman and Willems (1998) regarding the closedness of the system behavior set $\mathcal{B}$.

Proposition 4. If $(u_i, y_i) \in \mathcal{B}$ for $i \in \mathbb{N}$ is a sequence converging to $(\tilde{u}, \tilde{y})$ in the sense of $L^p_{loc}$, then $(\tilde{u}, \tilde{y}) \in \mathcal{B}$.

The following definition introduces a subset of $\mathcal{T}_p$ that represents the input–output pairs that perform the constrained rest-to-rest transition with a transition time less or equal than $M$.

Definition 4. Given a real number $M > 0$, the set of constrained rest-to-rest transitions with transition time bounded by $M$ is given by $\mathcal{T}^M_p = \{ (u, y) \in \mathcal{T}_p : T_p(u, y) \leq M \}$.

The following proposition shows that $\mathcal{T}^M_p$ is compact in the sense of $L_1$.

Proposition 5. Given any sequence of input–output pairs $(u_i, y_i) \in \mathcal{T}^M_p$, there exists a subsequence $(u_{i_k}, y_{i_k})$ and a pair $(u, y) \in \mathcal{T}^M_p$, such that

$$\lim_{k \to \infty} \int_0^M \left( |u - u_{i_k}| + |y - y_{i_k}| \right) \, dt = 0.$$
Proof. Define the functional
\[ T_f(u, y) = \inf \left\{ t_f \geq 0 : \forall (a, b) \in (t, +\infty) : \int_a^b \left| u - \frac{y_f}{H(0)} \right| dt = 0, \int_a^b |y - y_f| dt = 0 \right\}. \]

First of all we prove that \( T_f \) is a lower semicontinuous functional. This is equivalent to checking that
\[ T_f^{-1}((c, +\infty)) = \left\{ (u, y) \mid (a, b) \subset (c, +\infty), \epsilon > 0 : \int_a^b \left| u - \frac{y_f}{H(0)} \right| dt \geq \epsilon \right\} \]
is open, since \( T_f^{-1}(c, +\infty) \) is open and the complementary set \( \bar{T}_f^{-1}(-\infty, c) \) is closed. Consider the set \( \bar{T}_f^{-1}(c, +\infty) \), which can be written as \( \bar{T}_f^{-1}(c, +\infty) = E \cap B \cap \bar{T}_f^{-1}(-\infty, M) \), where \( E \) is a closed set in \( (u, y) \subset \mathcal{B} \cap \bar{T}_f^{-1}(-\infty, M) \).

Consider the following notation (see Polderman and Willems (1998), page 35) for the multiple integral of a function \( u(t) \). Define \( (f^{(i)}(u)) = u(t) \), and \( \forall i > 0, i \in \mathbb{N}, (f^{(i)}(u)) = \int_0^t (f^{(i-1)}(u))du. \)

Lemma 2. Let be given a function \( u(t) : \mathbb{R} \to \mathbb{R} \) and real numbers \( a < c < b \), then
\[ \int_a^b \int_0^t (f^{(i)}(u)(t))dt = 0 \text{ if and only if } \forall i > 0, i \in \mathbb{N}, (f^{(i)}(u)(t)) = c_0 + c_1t + c_2t^2 + \cdots + c_{i-1}t^{i-1}, \]
and
\[ (b) \int_a^b f^{(i)}(u)(t)dt = 0 \text{ and } \forall \in [a, b], (f^{(i)}(u)(t)) = p(t), \] where \( p(t) \) is a polynomial of degree \( n - 1 \), then \( \forall \in [a, b], (f^{(i)}(u)(t)) = p(t) \).

The proof is omitted for brevity.

Proposition 7. The optimal pair \((u^*, y^*)\) is essentially unique, i.e. if \((u, y) \in \mathcal{P}_f \) and \( T_f(u, y) = t_f^* \) then
\[ \int_0^{+\infty} \left( |u(t) - u^*(t)| + |y(t) - y^*(t)| \right) dt = 0. \]

Proof. Let \((u, y)\) be an input–output pair such that \( T_f(u, y) = t_f^* \) then all convex linear combinations of the form \((u_i, y_i) = (1 - \lambda)(u, y) + \lambda(u^*, y^*)\), satisfy \( T_f(u_i, y_i) = t_f^* \). As a consequence of
\[ \int_0^{+\infty} \min \left\{ |u(t) - u|^2, |y(t) - y^2| \right\} dt = 0, \]
implies that
\[ \int_0^{+\infty} \left( |u(t) - u^*(t)| + |y(t) - y^*(t)| \right) dt = 0. \]

The pair \((u, y)\) is a weak solution of (2) and satisfies e.a. 

\[ B[u(t)] = A[y(t)] + p(t). \]

where \( B[u] = \sum_{i=0}^n b_i(f^{(n-i+1)}(u)), A[y] = \sum_{i=0}^m a_i(f^{(n-i+1)}(y)) \) and \( p(t) \) is a suitable polynomial of degree not exceeding \( n \). Let \( U = \{ t \in \mathbb{R} : |u| \leq |y| \} \) and \( \mathcal{B} = U \cap \mathcal{Y} = \emptyset \) and \( U \cap \mathcal{Y} = \emptyset \). By Lebesgue integration theory, there exist countable closed intervals \( U_i, Y_i \) such that
\[ U \subset \bigcup_{i} U_i, Y \subset \bigcup_{i} Y_i, \left( \bigcup_{i} U_i \right) \cap \left( \bigcup_{i} Y_i \right) = \emptyset, \]
and the intervals are ordered according to \( U_i < Y_i < U_{i+1} < Y_{i+1} \), when \( \leq \) denotes the relation of left to right precedence between nonoverlapping intervals, that is \( a_i, b_1 \leq a \) when \( b_1 < a \). By (53), \( B[u](t) = 0 \) by part (b) of Lemma 2, there exist polynomials \( u_i \) of degree not exceeding \( n \) such that \( B[u_i](t) = u_i(t) \), \( Y_i \subset U_i \). In the same way, there exist polynomials \( y_i \) such that \( A[y_i](t) = y_i(t) \), \( Y_i \subset U_i \). From (54), it follows that \( \forall \in U_i, A[y_i](t) = B[u_i](t) = u_i(t) + p(t) \) and \( \forall \in U_i, B[u_i](t) = A[y_i] + p(t) = y_i(t) + p(t) \). Therefore, in each interval \( U_i, Y_i \), \( A[y_i](t) \) and \( B[u_i](t) \) are polynomials of degree less or equal than \( n \). Consider the two consecutive intervals \( U_i, Y_i \). Since \( \int_{U_i} f(y)(t)dt = 0 \) and \( A[y](t) \) is a polynomial in interval \( U_i \), then, by part (b) of Lemma 2, function \( A[y](t) \) must be equal to the same polynomial in interval \( Y_i \) that \( \forall \in U_i \cup Y_i, A[y](t) = u_i(t) + p(t) = y_i(t) \). Analogously, \( \int_{U_{i+1}} f(u)(t)dt = 0 \), then \( \forall \in U_{i+1}, B[u](t) = u_{i+1}(t) = y_{i+1}(t) + p(t) \). By equating the two different expressions for \( y_i(t) \), it follows that \( u_i = u_{i+1} \) for all \( i \). Hence, there exists one (unique) polynomial \( p_o \) such that
\[ B[u](t) = p_o(t), \forall t \in \mathbb{R}. \]
Likewise, there exists one (unique) polynomial $p_r$ satisfying
\[ A[y](t) = p_r(t), \quad \forall t \in \mathbb{R}. \]  
(56)

As a consequence of (55), by Theorem 3.2.4 of Polderman and Willems (1998), it follows that, almost everywhere, $\hat{u}$ can be expressed as a linear combination of the modes $m_i^2(t)$ associated to the zeros of (1) plus a constant term $c_0$, i.e., $\hat{u}(t) = c_0 + \sum_{i=1}^{n} c_i m_i(t)$. Since, $\forall t \geq t^*$, $\hat{u}(t) = u(t) - u^*(t) = 0$ (in fact the two functions reach the same final value), $c_0 = 0$, for $i = 0, \ldots, m$; then $\hat{u} = 0$ and $u(t) = u^*(t)$ almost everywhere, i.e., $\int_0^{\infty} |u(t) - u^*(t)|dt = 0$. In the same way, using relation (56) it follows that $\int_0^{+\infty} |y(t) - y^*(t)|dt = 0$. \hfill \Box

**Proposition 8.** Given a sequence of functions $(u_i, y_i) \in T_p$ if $\lim_{i \to +\infty} T_i(u_i, y_i) = t^*$, then $u_i \to u^*$, $y_i \to y^*$ in the sense of $L_1$ and $T_\gamma(u^*, y^*) = t^*$.

**Proof.** There exists a sufficiently large $M$ such that $(u_i, y_i) \in T^M_p$ for all $i \in \mathbb{N}$. As shown in the proof of Proposition 5, $T^M_p$ is a compact set, so that it is possible to find a convergent subsequence of pairs $(u_i, y_i)$ and its limit be denoted by $(\bar{u}, \bar{y})$. Hence $T_\gamma(\bar{u}, \bar{y}) = t^*$, and by Proposition 7, it follows that $\int_0^{+\infty} |u_i(t) - u^*(t)| + |y_i(t) - y^*(t)|dt = 0$. To prove that $(u_i, y_i)$ converges to $(u^*, y^*)$ assume by contradiction that it does not. Then, there exists an $\epsilon > 0$ such that $\forall i > 0$, $\exists i > l > 1: \int_0^{M} |u_i(t) - u^*(t)| + |y_i(t) - y^*(t)|dt > \epsilon$. Since $T^M_p$ is compact, it is possible to extract from the sequence with indexes $i_l, i_{l+1}, \ldots, \infty$ a convergent subsequence, whose limit is denoted by $(u_2, y_2)$, such that $T_\gamma(u_2, y_2) = t^*$ and $\int_0^{+\infty} |u_2(t) - u^*(t)| + |y_2(t) - y^*(t)|dt < \epsilon$, which contradicts Proposition 7. \hfill \Box

**Appendix B. Lemmas used in the proof of Theorem 4**

**Lemma 3.** Consider system $\Sigma (1)$, set $T > 0$, $t_0 \in \mathbb{R}$ and consider an input–output pair $(u, y) \in B$ for which $u(t) = \frac{\partial y}{\partial t}$, $\forall t \geq t_0$ and $y(t) = \int_{t_0}^{t} h(t - \tau)u(\tau)d\tau$ satisfies $y(t_0 + KT) = y_k$, for $k = 0, \ldots, n-1$. Moreover, assume that the distinct roots $p_1, \ldots, p_\ell$ of the polynomial $s^\ell + a_{\ell-1}s^{\ell-1} + \cdots + a_0$ satisfy $p_i \neq p_j, \forall \ell, r = 1, \ldots, l, \forall k \in \mathbb{Z} \neq \{0\}$ where $j$ denotes the imaginary unit. Then $y(t) = y_k$, $\forall t \geq t_0$.

The proof is based on the properties of the generalized Vandermonde matrix. This proof and those of the next two technical Lemmas have been omitted for sake of brevity.

**Lemma 4.** Consider system $\Sigma (1)$ and an input–output pair $(u, y) \in B$ for which $u(t)$ is constant in the intervals $[kT, (k + 1)T]$, $\forall k \in \mathbb{Z}$ and $y(t) = \int_{t_0}^{t} h(t - \tau)u(\tau)d\tau$ satisfies $y(t) = y_k$, $\forall k \in \mathbb{Z}$. Then $\forall t \in \mathbb{R}$
\[ y(t) - y_k \leq T \left( \|h(0^+\|) + \|Dh(\cdot)\|_{1} \|u(t)\|_\infty \right), \forall k \in \mathbb{Z}. \]  
(57)

**Lemma 5.** Consider system $\Sigma (1)$. Given $\epsilon > 0$, there exist two positive constants $M_u, M_y$, such that for any vector $\mathbf{x} = [z_0, z_1, \ldots, z_{n-1}]^T$ $\in \mathbb{R}^n$ and any $u, a \in \mathbb{R}$, there exists an input–output pair $(\hat{u}(t), \hat{y}(t)) \in B \cap C^0$ such that
\[ (1) \; \hat{u}(t) = 0, \forall t \leq a \text{ and } \hat{u}(t) = u, \forall t \geq a + \epsilon; \]
\[ (2) \; \hat{y}(t) = 0, \forall t \leq a, \hat{y}(a + \epsilon) = z_0, \quad D\hat{y}(a + \epsilon) = z_1, \ldots, \quad D^{n-1}\hat{y}(a + \epsilon) = z_{n-1} - 1; \]
\[ (3) \; \|\hat{u}\|_\infty \leq M_u(\|\mathbf{x}\| + |u|), \|\hat{y}\|_\infty \leq M_y(\|\mathbf{x}\| + |u|). \]

**References**


From 2005, Luca Consolini has been a postdoc at Dipartimento di Ingegneria dell’Informazione at the University of Parma, Italy. He was born in Parma in 1976. In 2000 he obtained the laurea cum laude in electronic engineering at the University of Parma. In 2005 he received the Ph.D. at the same University under the supervision of Prof. Aurelio Piazzì. In 2001 and 2002 he has been a visiting scholar at the University of Toronto, Canada, under the supervision of Prof. Manfredi Maggiore. Since 2000 he has collaborated actively with Prof. Mario Tosques, professor of mathematical analysis at the University of Parma. His main research topics are dynamic inversion for nonlinear systems, tracking and path following, formation control and time-optimal control. Since 2005 he has been teaching the course of “Digital Control” at the faculty of Engineering of the University of Parma.

Aurelio Piazzì received the Laurea degree in nuclear engineering in 1982 and the Ph.D. degree in system engineering in 1987, both from the University of Bologna, Italy. From 1992 he has been affiliated with the University of Parma, Italy where he is full professor of Control Systems. His main research interests are in control theory, autonomous robotics, and mechatronics systems. His recent research activities have focused on feedforward/feedback methods for the control of uncertain systems and for the autonomous navigation of wheeled robots and vehicles. He has been the scientific coordinator of many industry research programs and in the years 2007–2008, in collaboration with RFI Ferrovie dello Stato Italia, he directed the research project PAVSYS (Pantograph Automatic Vision-based Inspection System) devoted to the in-service diagnosis of railway pantographs. He is a member of IEEE and SIAM. His research findings have been published in over 100 scientific papers in international journals and conference proceedings.