

A Global Optimization Approach to Scalar H_2/H_∞ Control

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Various approaches to the mixed H_2/H_∞ control problem are proposed in the literature. The solution is often obtained by means of classical techniques (i.e. solving a Riccati problem) or by means of convex optimization procedures. With the aim to synthesize a fixed-structure controller for an uncertain plant, a global optimization approach to a single-input single-output (SISO) H_2/H_∞ problem is proposed. In particular, robust stability is guaranteed and a nominal quadratic cost index is minimized by solving an equivalent nonconvex semi-infinite optimization problem. The resulting design method that relies on a recently devised genetic/interval algorithm is tested by designing low-order controllers for two example plants.

Keywords: Genetic algorithms; H_∞ control; Interval algorithms; Optimization; Quadratic cost indices

1. Introduction

The mixed H_2/H_∞ approach to finite-dimensional linear time-invariant control systems design has been variously investigated in the recent literature [1,4]. As can be evinced also from the Kwakernaak's tutorial paper [15], the standard H_∞ robust control leaves degrees of freedom in the controller design. For this reason the H_∞ problem is often coupled to the minimization of an H_2 (quadratic) nominal cost index.

A procedure for discrete-time scalar systems is proposed by Sznaier [23] by means of a sequence of convex

optimization problems which finally yields a suboptimal finite-dimensional controller. Convex optimization is also used in [22], which addresses a more general multiobjective H_2/H_∞ design problem for multivariable plants. In recent times, the H_2/H_∞ approach has attracted the attention of researchers for solving a classical problem: the PID tuning for an uncertain plant. The reason of such interest lies in potential industrial applications of the new methodologies. The papers [3] and [24] belong to this current.

In this paper, in the context of continuous-time, scalar (single-input single-output) systems, a solution to an H_2/H_∞ fixed-structure controller design problem is proposed via a global optimization approach. Preliminary results were presented in [8] for a plant with multiplicative uncertainties and comparisons with the PID controller design method of Chen *et al.* [3] were made. In this work a more general approach is considered by adopting the feedback generalized plant scheme of [23]. Specifically, an arbitrary fixed-structure controller is designed to globally minimize a nominal H_2 cost, subject to robust closed-loop stability secured through an H_∞ constraint and a reformulation of the Liénard–Chipard criterion. A salient feature of this approach lies in the use of a hybrid two-level (genetic/interval) algorithm. This approach permits finding an estimated global minimizer that is feasible with certainty, i.e. the synthesized controller definitely guarantees the fulfillment of the robust stability and, at the same time, provides a good estimate of the H_2 global minimum. Focusing on multiplicative uncertainty plant models, an alternative approach to scalar H_2/H_∞ fixed-structure controller design is provided by Krohling [14] using a homogeneous two-level (genetic/genetic) algorithm. This approach, due to the

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fact that a stochastic technique (the genetic routine) cannot guarantee the complete fulfillment of a semi-infinite constraint (the H_∞ one for the case at hand), though quite effective on many instances, cannot ensure with certainty that the estimated optimal controller provides closed-loop stability for all the members of the uncertain plant family. On the contrary, the approach proposed in this paper using a deterministic global technique (an interval procedure) makes sure the fulfillment of the H_∞ constraint [19].

In Section 2, we pose and solve the H_2/H_∞ problem. In particular this section shows how to convert the H_∞ constraint into a semi-infinite inequality over a real bounded interval. Then, the resulting semi-infinite optimization problem is reduced to a finite bound-constrained problem whose estimated global solution can be obtained by using a recently devised genetic/interval algorithm [7,9]. Section 3 proposes two control problems, one associated with a second-order plant and the other with a two-mass-spring plant (in [23], the latter was already examined using the H_2/H_∞ framework). For both examples it is shown how to obtain the corresponding generalized plant (that is not the "actual" plant) in order to set up the H_2/H_∞ problem. Computational results, simulations, and comparisons are included. Final comments are reported in Section 4.

2. The optimal H_2/H_∞ controller design

The given generalized plant Σ and the output-to-input dynamic feedback $C(s; \mathbf{x})$ are shown in Fig. 1. Two inputs (ω_∞ and ω_2) are exogenous scalar disturbances while the third one is the system control input u . In the same way Σ has two generalized controlled outputs (ζ_∞ and ζ_2) and one sensed (regulated) output y . The fixed-structure, rational, proper controller $C(s; \mathbf{x})$ is parametrized by the vector $\mathbf{x} := [x_1, x_2, \dots, x_l]^T \in \mathcal{X} = [x_1^-, x_1^+] \times [x_2^-, x_2^+] \times \dots \times [x_l^-, x_l^+] \subseteq \mathbb{R}^l$. The transfer functions between ω_∞ and ζ_∞ and between ω_2 and ζ_2 are denoted as $T_{\zeta_\infty \omega_\infty}(s; \mathbf{x})$

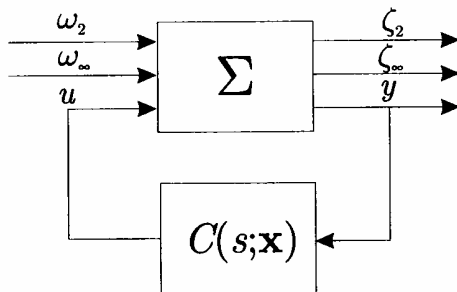


Fig. 1. The generalized plant and the fixed-structure controller $C(s; \mathbf{x})$.

and $T_{\zeta_2 \omega_2}(s; \mathbf{x})$, respectively. The optimal H_2/H_∞ controller design problem aims to find an \mathbf{x}^* such that the feedback controller $C(s; \mathbf{x}^*)$ internally stabilizes the plant while minimizing the H_2 norm $\|T_{\zeta_2 \omega_2}(s; \mathbf{x}^*)\|_2$ subject to the H_∞ constraint $\|T_{\zeta_\infty \omega_\infty}(s; \mathbf{x}^*)\|_\infty \leq 1$. The rationale behind this problem formulation is to find an optimal fixed-structure controller that minimizes a nominal H_2 cost, subject to the robust closed-loop stability constraint. The transfer functions $T_{\zeta_\infty \omega_\infty}(s; \mathbf{x})$ and $T_{\zeta_2 \omega_2}(s; \mathbf{x})$ of the generalized plant depend, respectively, on the unstructured uncertainty associated with the actual plant and on the chosen quadratic index, e.g. the integral of the squared regulated error in response to a unit-step reference (cf. Section 3.1).

The nominal characteristic polynomial associated with the internal closed-loop stability be denoted by $\xi(s; \mathbf{x}) = \sum_{i=0}^n \xi_i(\mathbf{x})s^i$ and the associated i th-order Hurwitz determinant by $H_i(\mathbf{x})$. To avoid degenerate system configurations the following assumptions are introduced.

Assumption 1. $T_{\zeta_\infty \omega_\infty}(s; \mathbf{x})$ and $T_{\zeta_2 \omega_2}(s; \mathbf{x})$ are strictly proper fixed-order rational functions for any $\mathbf{x} \in \mathcal{X}$. □

Assumption 2. The leading coefficient $\xi_n(\mathbf{x})$ of the nominal characteristic polynomial is always positive for any $\mathbf{x} \in \mathcal{X}$. □

The following reformulation of the Liénard–Chipard criterion [6, p. 221] can be fruitfully used to ensure the internal stability of the feedback system of Fig. 1.

Property 1. The fixed-structure controller $C(s; \mathbf{x})$ internally stabilizes the closed-loop systems if and only if ($n \geq 3$):

$$\xi_0(\mathbf{x}) > 0; \quad \xi_1(\mathbf{x}) > 0, \xi_3(\mathbf{x}) > 0, \dots, \xi_v(\mathbf{x}) > 0; \\ H_{n-1}(\mathbf{x}) > 0, H_{n-3}(\mathbf{x}) > 0, \dots, H_q(\mathbf{x}) > 0,$$

with $v := n - 1$, $q := 3$ if n is even and $v := n - 2$, $q := 2$ if n is odd. □

The numerator and denominator polynomials of the ω -rational function $|T_{\zeta_\infty \omega_\infty}(j\omega; \mathbf{x})|^2 = T_{\zeta_\infty \omega_\infty}(j\omega; \mathbf{x})T_{\zeta_\infty \omega_\infty}(-j\omega; \mathbf{x})$ be denoted by $\beta(\omega; \mathbf{x})$ and $\alpha(\omega; \mathbf{x})$, respectively.

Property 2. There exists a finite positive $\tilde{\omega}(\mathbf{x}) \in \mathbb{R}^+$, depending on $\mathbf{x} \in \mathcal{X}$, for which the H_∞ constraint

$$\|T_{\zeta_\infty \omega_\infty}(s; \mathbf{x})\|_\infty \leq 1 \tag{1}$$

is equivalent to

$$\beta(\omega; \mathbf{x}) - \alpha(\omega; \mathbf{x}) \leq 0 \quad \forall \omega \in [0, \tilde{\omega}(\mathbf{x})]. \tag{2}$$

Proof. By virtue of the H_∞ norm definition, (1) is clearly equivalent to the inequality

$$|T_{\zeta_x \omega_x}(j\omega; \mathbf{x})| \leq 1 \quad \forall \omega \in [0, +\infty).$$

and also to

$$|T_{\zeta_x \omega_x}(j\omega; \mathbf{x})|^2 = \frac{\beta(\omega; \mathbf{x})}{\alpha(\omega; \mathbf{x})} \leq 1 \quad \forall \omega \in [0, +\infty). \quad (3)$$

Inequality (3) can be arranged as

$$\beta(\omega; \mathbf{x}) - \alpha(\omega; \mathbf{x}) \leq 0 \quad \forall \omega \in [0, +\infty). \quad (4)$$

Assumption 1 and the definition of the polynomials $\alpha(\omega; \mathbf{x})$ and $\beta(\omega; \mathbf{x})$ permit writing:

- (a) $\deg(\alpha(\omega; \mathbf{x})) > \deg(\beta(\omega; \mathbf{x}))$;
- (b) the leading coefficient of $\alpha(\omega; \mathbf{x})$ is positive;
- (c) both $\alpha(\omega; \mathbf{x})$ and $\beta(\omega; \mathbf{x})$ have only even powers of ω .

By virtue of the above statements (a) and (b) the following equation is satisfied:

$$\lim_{\omega \rightarrow \infty} [\beta(\omega; \mathbf{x}) - \alpha(\omega; \mathbf{x})] = -\infty,$$

so that $\beta(\omega; \mathbf{x}) - \alpha(\omega; \mathbf{x})$ reaches a finite maximum over $[0, +\infty)$. Therefore, the semi-infinite inequality (4) is equivalent to

$$\beta(\omega^*(\mathbf{x}); \mathbf{x}) - \alpha(\omega^*(\mathbf{x}); \mathbf{x}) \leq 0 \quad (5)$$

with $\omega^*(\mathbf{x}) \in \mathbb{R}$ defined as

$$\omega^*(\mathbf{x}) := \arg \max \{ \beta(\omega; \mathbf{x}) - \alpha(\omega; \mathbf{x}) : \omega \in [0, +\infty) \}.$$

Inequality (5) is clearly equivalent to (2) if $\tilde{\omega}(\mathbf{x})$ is an upper bound of $\omega^*(\mathbf{x})$. To determine $\tilde{\omega}(\mathbf{x})$ we can safely suppose that $\omega^*(\mathbf{x}) > 0$. Indeed, if $\omega^*(\mathbf{x}) = 0$ then any positive value of $\tilde{\omega}(\mathbf{x})$ is an upper bound of $\omega^*(\mathbf{x})$. Let the polynomial $\beta(\omega; \mathbf{x}) - \alpha(\omega; \mathbf{x})$ be described by

$$p_{2m}(\mathbf{x})\omega^{2m} + p_{2m-2}(\mathbf{x})\omega^{2m-2} + \dots + p_2(\mathbf{x})\omega^2 + p_0(\mathbf{x})$$

with $m := \deg(\alpha(\omega; \mathbf{x}))/2$ and $p_{2m}(\mathbf{x}) < 0 \quad \forall \mathbf{x} \in \mathcal{X}$. The maximizer $\omega^*(\mathbf{x})$ must be a root of the derivative polynomial

$$\begin{aligned} \frac{\partial}{\partial \omega} (\beta(\omega; \mathbf{x}) - \alpha(\omega; \mathbf{x})) \\ = 2mp_{2m}(\mathbf{x})\omega^{2m-1} + (2m-2)p_{2m-2}(\mathbf{x})\omega^{2m-3} \\ + \dots + 2p_2(\mathbf{x})\omega \end{aligned}$$

or, excluding the zero root, of the polynomial

$$mp_{2m}(\mathbf{x})\omega^{2m-2} + (m-1)p_{2m-2}(\mathbf{x})\omega^{2m-4} + \dots + p_2(\mathbf{x}).$$

Hence $[\omega^*(\mathbf{x})]^2$ is a root of the η -polynomial

$$mp_{2m}(\mathbf{x})\eta^{m-1} + (m-1)p_{2m-2}(\mathbf{x})\eta^{m-2} + \dots + p_2(\mathbf{x}).$$

By using the classical Cauchy bound we infer

$$[\omega^*(\mathbf{x})]^2 < 1 + \max_{1 \leq j \leq m-1} \left\{ \frac{(m-j)|p_{2(m-j)}(\mathbf{x})|}{m|p_{2m}(\mathbf{x})|} \right\}.$$

Therefore, we finally define $\tilde{\omega}(\mathbf{x})$ as

$$\tilde{\omega}(\mathbf{x}) := \sqrt{1 + \max_{1 \leq j \leq m-1} \left\{ \frac{(m-j)|p_{2(m-j)}(\mathbf{x})|}{m|p_{2m}(\mathbf{x})|} \right\}}. \quad (6)$$

□

Remark 1. The previous proof does not only demonstrate the equivalence between (1) and (2) but is also fully constructive because it gives a definite expression for a proper $\tilde{\omega}(\mathbf{x})$.

As previously anticipated, the cost index to be minimized is given by the H_2 norm $\|T_{\zeta_2 \omega_2}(s; \mathbf{x})\|_2$ denoted in the following by $J(\mathbf{x})$ and computed, for example by means of the determinantal method of Katz [13] (cf. the Appendix).

By considering Properties 1 and 2, the addressed H_2/H_∞ fixed-structure controller design can be posed as the following global semi-infinite optimization problem:

$$\min_{\mathbf{x} \in \mathcal{X}} J(\mathbf{x}) \quad (7)$$

subject to

$$\xi_0(\mathbf{x}) > 0; \quad \xi_1(\mathbf{x}) > 0, \quad \xi_3(\mathbf{x}) > 0, \quad \dots, \quad \xi_v(\mathbf{x}) > 0; \quad (8)$$

$$H_{n-1}(\mathbf{x}) > 0, \quad H_{n-3}(\mathbf{x}) > 0, \quad \dots, \quad H_q(\mathbf{x}) > 0; \quad (9)$$

$$\beta(\omega; \mathbf{x}) - \alpha(\omega; \mathbf{x}) \leq 0 \quad \forall \omega \in [0, \tilde{\omega}(\mathbf{x})]. \quad (10)$$

If the fixed-structure controller $C(s; \mathbf{x})$ that has to be pertinently chosen by the control engineer admits robust closed-loop stability, then problem (7)–(10) has a solution and evidently the constraints (8) and (9) cannot be active at the global minimum. Hence, there exists a sufficiently small positive $\epsilon \in \mathbb{R}^+$ such that problem (7)–(10) is equivalent to the following one:

$$\min_{\mathbf{x} \in \mathcal{X}} J(\mathbf{x}) \quad (11)$$

subject to

$$\xi_0(\mathbf{x}) \geq \epsilon; \quad \xi_1(\mathbf{x}) \geq \epsilon, \quad \xi_3(\mathbf{x}) \geq \epsilon; \quad \dots, \quad \xi_v(\mathbf{x}) \geq \epsilon; \quad (12)$$

$$H_{n-1}(\mathbf{x}) \geq \epsilon, \quad H_{n-3}(\mathbf{x}) \geq \epsilon, \quad \dots, \quad H_q(\mathbf{x}) \geq \epsilon; \quad (13)$$

$$\beta(\omega; \mathbf{x}) - \alpha(\omega; \mathbf{x}) \leq 0 \quad \forall \omega \in [0, \tilde{\omega}(\mathbf{x})]. \quad (14)$$

It is worth noting that again the constraints (12) and (13) are not active on the optimal solution (typically the active constraint is (14), cf. the example in Section 3.2) and on the practical viewpoint choosing a correct ϵ is not an issue: just select the smallest positive real compatible with the implemented precision of computations.

The optimization problem (11)–(14), which is of semi-infinite type due to the presence of the semi-infinite constraint (14), generally involves nonlinear and nonconvex functions: the $J(\mathbf{x})$ as well as the constraint functions. Hence, conventional algorithms such as those proposed in [12,20] and could be easily trapped in local minima or even end up with “optimal solutions” that are not feasible, i.e. “solutions” not satisfying the semi-infinite constraint(s). With the aim of determining an estimated global minimizer \mathbf{x}^* to problem (11)–(14), an effective alternative is the global optimization algorithm proposed by the authors [7,9] for standard semi-infinite problems. It is an hybrid algorithm based on a genetic algorithm at the upper level and on an interval procedure at the lower level to handle the semi-infinite constraints.

By using a penalty method, problem (11)–(14) is transformed into the following (finite) minimization problem:

$$\min_{\mathbf{x} \in \mathcal{X}} \left\{ J(\mathbf{x}) + \sum_{i=1}^{n+1} \Phi(\sigma_i(\mathbf{x})) \right\}. \quad (15)$$

The penalty function is defined by

$$\Phi(\sigma) := \begin{cases} 0 & \text{if } \sigma \leq 0, \\ M - M(T - \sigma)^2/T^2 & \text{if } 0 < \sigma < T, \\ M & \text{if } \sigma \geq T \end{cases}$$

and its arguments are defined accordingly to

$$\sigma_i(\mathbf{x}) := \begin{cases} \epsilon - \xi_0(\mathbf{x}) & \text{if } i = 1, \\ \epsilon - \xi_{2i-3}(\mathbf{x}) & \text{if } i = 2, 3, \dots, j, \\ \epsilon - H_{2i-n-1}(\mathbf{x}) & \text{if } i = j+1, \dots, n, \\ \max_{\omega \in [0, \tilde{\omega}(\mathbf{x})]} \{ \beta(\omega; \mathbf{x}) - \alpha(\omega; \mathbf{x}) \} & \text{if } i = n+1, \end{cases}$$

where $j = 1 + n/2$ if n is even while $j = (n+1)/2$ if n is odd. The positive parameters M and T used by the penalty function $\Phi(\sigma)$ must be sufficiently large and small respectively, depending on the chosen numerical accuracy.

The partially elitistic genetic algorithm of [17] can be used to solve the unconstrained problem (15). While computing the penalty terms $\Phi(\sigma_i(\mathbf{x}))$, $i = 1, \dots, n$ is straightforward, the term $\Phi(\sigma_{n+1}(\mathbf{x}))$ needs a special interval procedure, i.e. a deterministic algorithm which uses concepts of interval analysis [18] and converges with certainty within a prespecified tolerance. Essentially, it is based on a branch-and-bound technique used for exhaustive search on $[0, \tilde{\omega}(\mathbf{x})]$ and on the use of the so-called *inclusion functions* of interval analysis [11]. This procedure can be regarded as a variant of the interval positivity test (IPT) for multivariate functions exposed in [19]. Further details can be found in [7,9].

It should be made clear that the genetic/interval algorithm proposed to solve (11)–(14) does not provide a guaranteed global minimizer, but provides an estimate of the global minimizer that is feasible with certainty. As shown in [9], which also reports extensive results on test problems, this estimate can indeed be an excellent solution of the semi-infinite problem. Moreover, the deterministic and global nature of the interval procedure makes sure that the estimate satisfies with certainty the semi-infinite constraint(s). This feature of the hybrid algorithm renders the approach interesting to tackle other control design problems, e.g. the optimal worst-case output feedback [10].

3. Examples

Two examples are reported in the following subsections. For both of them two different fixed-structure controllers have been designed and their performances have been analyzed. The first example considers a second-order fractionary uncertain plant. It is known (see [26, pp. 227–228]) that such models can be used to appropriately describe systems with a changing number of right half-plane poles. The second example is derived from a well-known benchmark problem: the two-mass-spring plant proposed at the 1990 American Control Conference [25].

Both problems have been solved by using the genetic/interval algorithm presented in [7,9] which has been coded and compiled with GCC for MS Windows on a PC Pentium III 500 MHz.

3.1. Second-order plant

Let the plant be described by the fractionary uncertainty model \mathcal{P} given by

$$\mathcal{P} := \{ \tilde{\mathcal{P}}(s) : \tilde{\mathcal{P}}(s) := P(s)/[1 + \Delta W_2(s)], \|\Delta\|_\infty < 1 \} \quad (16)$$

with $P(s)$ the transfer function of the nominal plant, $W_2(s)$ the fixed stable weighting function, and Δ the scaling factor of $W_2(s)$. As usual, it is assumed that unstable poles of $P(s)$ cannot be cancelled by $1/[1 + \Delta W_2(s)]$. A unity-feedback control scheme is adopted. The fixed-structure, proper controller $C(s; \mathbf{x})$, with the design parameter vector \mathbf{x} belonging to the multi-interval $\mathcal{X} \subseteq \mathbb{R}^l$, must guarantee (a) the robust stability of the closed-loop system, (b) the minimization over \mathcal{X} of $\int_0^\infty e^2(t; \mathbf{x}) dt$ where $e(t; \mathbf{x})$ is the output tracking error to a unit-step reference signal evaluated for the nominal plant $P(s)$. Both requirements can be fulfilled by matching the problem with the general framework proposed in the previous section. Due to the uncertainty model (16), robust stability is obtained by imposing $T_{\zeta_x \omega_x}(s; \mathbf{x}) := W_2(s)/(1 + P(s)C(s; \mathbf{x}))$ [5, see p. 55]. Denoting by $E(s; \mathbf{x})$ the Laplace transform of $e(t; \mathbf{x})$, we can set $T_{\zeta_2 \omega_2}(s; \mathbf{x}) := E(s; \mathbf{x}) = 1/\{s[1 + P(s)C(s; \mathbf{x})]\}$. By virtue of the above definitions, the generalized controlled system can be represented as in Fig. 2.

The actual uncertain plant \mathcal{P} is defined by

$$\mathcal{P} := \left\{ \tilde{P}(s) : \tilde{P}(s) = \frac{1}{(s+4)(s+1+\delta)}, \|\delta\|_\infty < 2 \right\}. \quad (17)$$

The uncertain pole of model (17) can belong both to the left or to the right half-plane. Model (17) can be arranged in the general form (16) by imposing

$$P(s) := \frac{1}{(s+1)(s+4)}, \quad W_2(s) := \frac{2}{(s+1)}.$$

Two cases are considered for the fixed-structure controller:

1. PI control

$$C_1(s; \mathbf{x}) := \frac{x_1 + x_2 s}{s},$$

$$\mathbf{x} \in \mathcal{X} := [0, 100] \times [-500, 500].$$

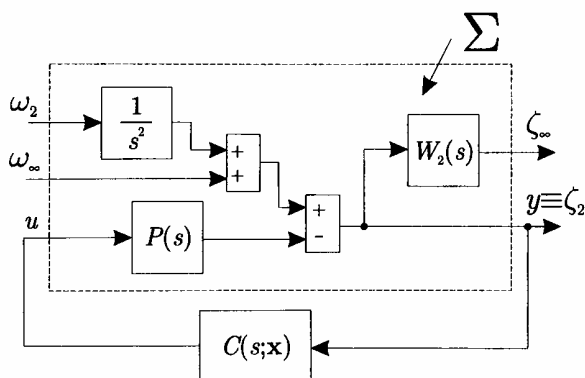


Fig. 2. Block diagram of the generalized controlled plant.

2. PID control

$$C_2(s; \mathbf{x}) := x_1 \frac{s^2 + 2x_2 x_3 s + x_3^2}{s(s+a)},$$

$$\mathbf{x} \in \mathcal{X} := [0, 5000] \times [0, 10] \times [0, 10],$$

where $a = 50$ is a "fast" pole introduced to obtain a proper controller.

PI controller. The genetic/interval algorithm has converged to the estimated global minimizer $\mathbf{x}^* = [19.52 \ 500.0]^T$ with optimal cost index $\int_0^\infty e^2(t; \mathbf{x}^*) dt = 0.105$ and with an average computation time of 2 s. It is worth noting that \mathbf{x}^* is on the boundary of \mathcal{X} . This corresponds to an almost pole-zero cancellation in the origin of the complex plane for the optimal controller $C_1(s; \mathbf{x}^*)$. The unit-step response for the nominal plant shown in Fig. 3 has an unacceptable oscillatory behaviour.

Marginal improvements on the cost index are obtained by enlarging \mathcal{X} but with a drawback of larger overshoots in the unit-step response. Thus, from a control engineering viewpoint, a PI controller is inconvenient for this kind of uncertain plant so that the next PID controller design is proposed.

PID controller. The evaluated optimal solution is $\mathbf{x}^* = [5000 \ 0.5683 \ 1.915]^T$ with nominal performance $\int_0^\infty e^2(t; \mathbf{x}^*) dt = 0.01456$ and an average computation time equal to 7 s. Again, the minimizer \mathbf{x}^* is on the boundary of \mathcal{X} and corresponds to the selection of the maximal allowable velocity constant. This solution provides a controller with two complex zeroes ($-1.088 \pm j1.576$). The nominal performance is definitively better than that obtained with the PI controller. The exhibited unit-step response, shown by the solid line in Fig. 3, is better too; it has a sharply damped oscillatory behaviour with a considerably smaller overshoot $M_p = 25.5\%$.

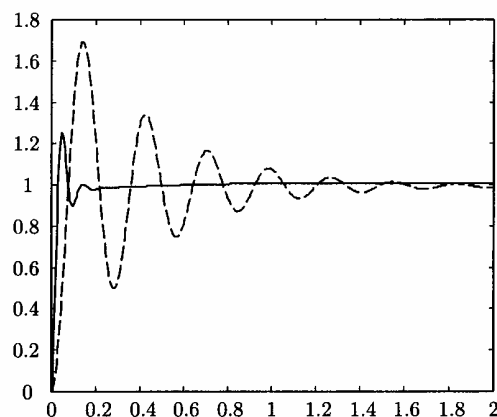


Fig. 3. Unit-step responses for the nominal plant, obtained with $C_1(s; \mathbf{x}^*)$ (dashed line) and $C_2(s; \mathbf{x}^*)$ (solid line).

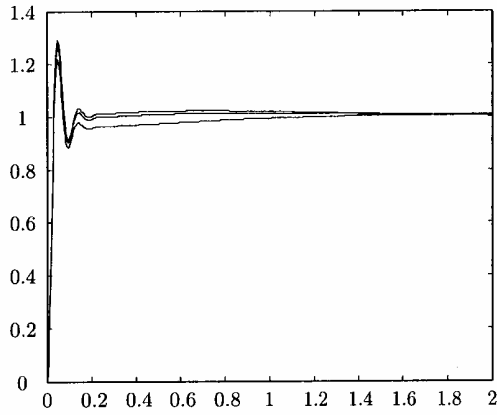


Fig. 4. Unit-step responses for $\tilde{P}_1(s)$, $\tilde{P}_2(s)$ and $\tilde{P}_3(s)$ obtained with $C_2(s; \mathbf{x}^*)$.

Figure 4 shows the step responses associated with the following three plants related to the uncertain model \mathcal{P} :

$$\tilde{P}_1(s) := \frac{1}{(s+4)(s+3)}, \quad (18)$$

$$\tilde{P}_2(s) := \frac{1}{s(s+4)}, \quad (19)$$

$$\tilde{P}_3(s) := \frac{1}{(s+4)(s-1)}. \quad (20)$$

Plants $\tilde{P}_1(s)$ and $\tilde{P}_3(s)$ belong to the boundary of \mathcal{P} , while $\tilde{P}_2(s)$ has the uncertain pole located at the origin of the complex plane. In spite of the diversity of the three plants, quite similar step responses have been obtained.

3.2. Two-mass-spring plant

The original version of this benchmark problem was first proposed in the 1990 at the American Control Conference [25] to verify the capability of controllers to reject disturbances in presence of plant uncertainties. In the following, we adopt the design specifications proposed by Sznaier [23]. The plant is the classical undamped two-mass-spring system shown in Fig. 5. Two unit masses ($m_1 = 1$, $m_2 = 1$) are coupled by a spring whose elastic constant k is uncertain: $k \in (0.5, 2)$. The controller has to reject a white noise disturbance ω_2 acting on the second mass by means of a force u working on the first mass. Differently from Szainer, we have adopted a continuous-time fixed-structure controller $C(s; \mathbf{x})$ parametrized

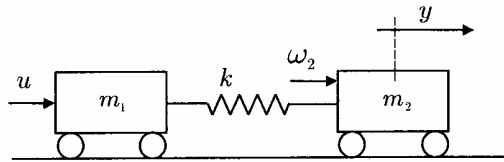


Fig. 5. The two-mass-spring plant.

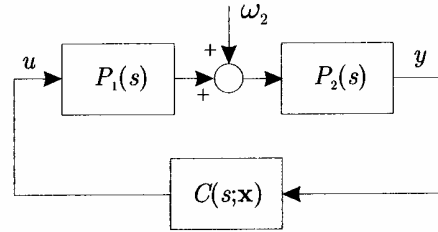


Fig. 6. The closed-loop system.

by $\mathbf{x} \in \mathcal{X}$. The resulting closed-loop system is shown in Fig. 6, where

$$P_1(s) = \frac{k}{m_1 s^2 + k}, \quad (21)$$

$$P_2(s) = \frac{m_1 s^2 + k}{s^2 [m_1 m_2 s^2 + k(m_1 + m_2)]}. \quad (22)$$

Sznaier proposed in [23] to model the spring constant as $k := k_0 + \Delta$, where $k_0 = 1.25$ and $|\Delta| < 1/\gamma := 0.75$. Disturbance rejection is achieved by minimizing, for the nominal plant (i.e. $k := k_0$), the RMS value of the control signal u in response to a white noise ω_2 acting on m_2 or, equivalently, by minimizing the H_2 -norm of the transfer function between ω_2 and u ,

$$T_{\zeta_2 \omega_2}(s; \mathbf{x}) = \frac{C(s; \mathbf{x})(m_1 s^2 + k_0)}{s^2 [m_1 m_2 s^2 + k_0(m_1 + m_2)] - k_0 C(s; \mathbf{x})}. \quad (23)$$

The *small-gain theorem* has been applied to ensure the robust stability of the closed loop system for any $\|\Delta\|_\infty < 1/\gamma$, where Δ is a stable unstructured uncertainty. The closed-loop system can be arranged in the usual standard form shown in Fig. 7, where the uncertainty Δ has been pulled out. According to the small-gain theorem for any stable Δ such that $\|\Delta\|_\infty < 1/\gamma$, $\gamma > 0$, the closed-loop system is robustly stable if $M(s; \mathbf{x})$ is stable and $\|M(s; \mathbf{x})\|_\infty \leq \gamma$ or, equivalently, defining $T_{\zeta_\infty \omega_\infty}(s; \mathbf{x}) := M(s; \mathbf{x})/\gamma$, if $\|T_{\zeta_\infty \omega_\infty}(s; \mathbf{x})\|_\infty \leq 1$. For the two-mass-spring system

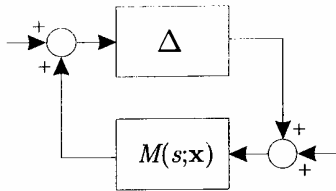


Fig. 7. The standard form of the uncertain closed-loop system.

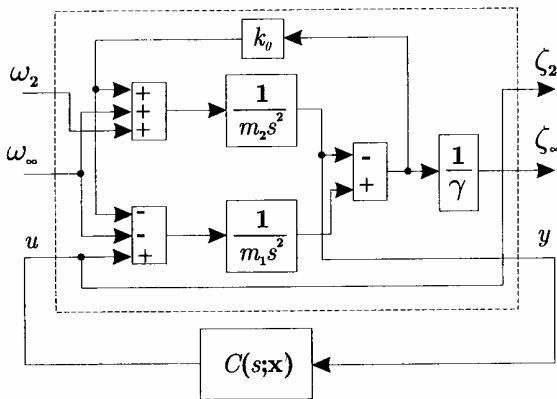


Fig. 8. The generalized controlled plant for the two-mass-spring system.

it holds that

$$T_{\zeta_2 \omega_\infty}(s; \mathbf{x}) = \frac{C(s; \mathbf{x}) - s^2(m_1 + m_2)}{s^2[m_1 m_2 s^2 + k_0(m_1 + m_2)] - k_0 C(s; \mathbf{x}) \gamma} \cdot 1 \quad (24)$$

The resulting generalized controlled plant is shown in Fig. 8.

In the following, two fixed-structure controllers have been designed: a second-order controller is compared with a third-order controller. Both controllers are biproper.

Second-order controller. The following second-order structure has been chosen:

$$C_1(s; \mathbf{x}) := x_1 \frac{(s + x_2)(s + x_3)}{s^2 + 2x_4 x_5 s + x_5^2}, \quad (25)$$

where $\mathbf{x} := [x_1 \ x_2 \ x_3 \ x_4 \ x_5]^T \in \mathcal{X} := [0.01, 10] \times [-2, 2] \times [-2, 2] \times [0.6, 1] \times [0.01, 5]$. The genetic/interval algorithm has converged, with an average computation time of 7 min 51 s, to the estimated global minimizer $\mathbf{x}^* = [4.7068 \ 6.1646 \cdot 10^{-3} \ -0.13685 \ 0.64392 \ 1.9641]^T$ with nominal cost index $J(\mathbf{x}^*) = 1.18380$. The amplitude Bode plot of the transfer function $T_{\zeta_2 \omega_\infty}(s; \mathbf{x}^*)$ shows that the semi-infinite constraint is active at the optimal solution (see Fig. 9).

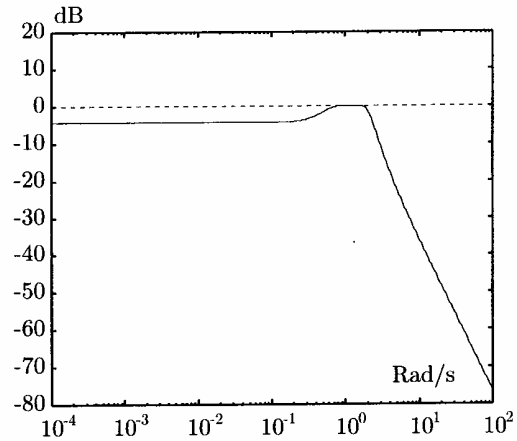


Fig. 9. Amplitude Bode plot of the transfer function $T_{\zeta_2 \omega_\infty}(s; \mathbf{x}^*)$.

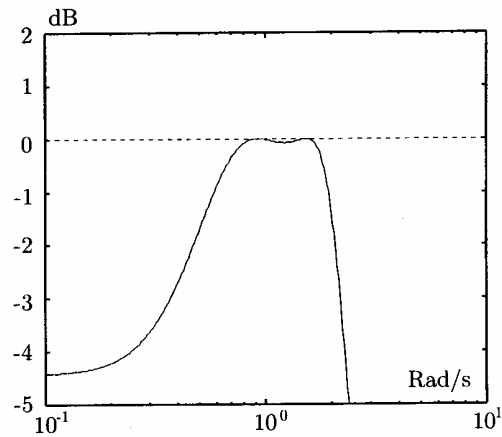


Fig. 10. A detail of $|T_{\zeta_2 \omega_\infty}(j\omega; \mathbf{x}^*)|$.

In Fig. 10, a detail of the same plot reveals that the constraint limit value is reached at two different frequencies.

The closed-loop system has a dominant pole placed at -0.006671 . Its position is practically independent of the uncertain k . The root loci for the nominal plant and for the two extreme plants ($k=0.5$ and $k=2$) are reported in Fig. 11.

In Fig. 12 the response of the nominal closed-loop system to a unit impulse is shown. The response peak value (close to 1) is much smaller than the corresponding one found by Sznaier (close to -15). This is in accordance with the cost index (equal to 1.18380) of the proposed second-order controller compared with the cost index found in [23] (equal to 22.6493).

Third-order controller. To further reduce the nominal cost index, the following third-order controller

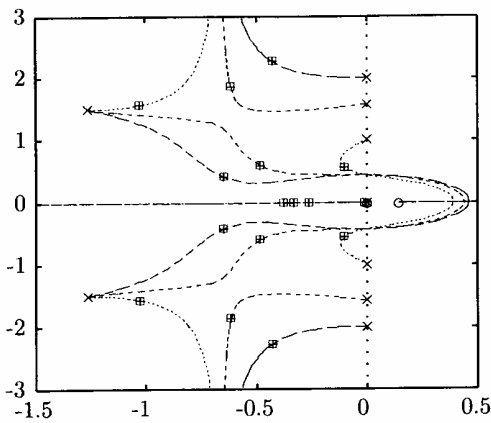


Fig. 11. Root loci for $C_1(s; \mathbf{x}^*)$: squares indicate the position of the closed-loop poles.

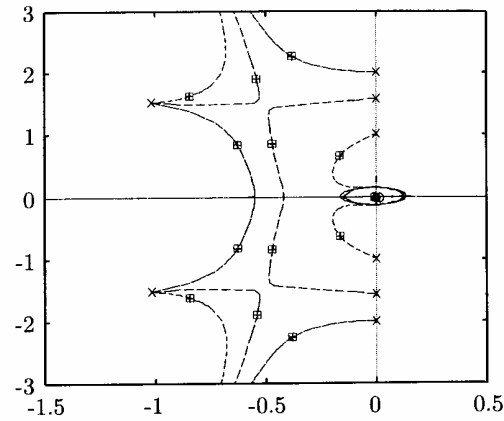


Fig. 13. Root loci for $C_2(s; \mathbf{x}^*)$: squares indicate the positions of the closed-loop poles.

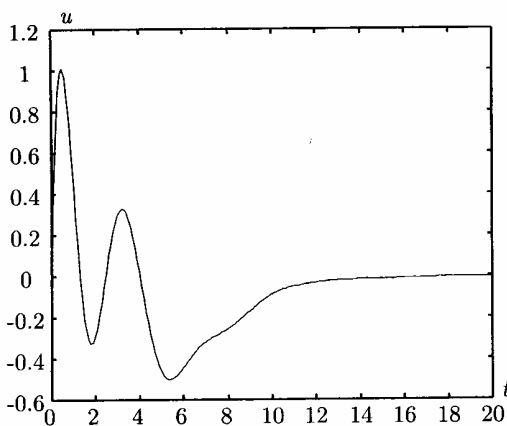


Fig. 12. Nominal unit impulse response of the closed-loop system.

has been introduced:

$$C_2(s; \mathbf{x}) := x_1 \frac{(s + x_2)(s + x_3)(s + x_4)}{(s + x_5)(s^2 + 2x_6x_7s + x_7^2)}, \quad (26)$$

where $\mathbf{x} := [x_1 \ x_2 \ x_3 \ x_4 \ x_5 \ x_6 \ x_7]^T \in \mathcal{X} := [0.01, 15] \times [-2, 2] \times [-2, 2] \times [-20, 20] \times [0.0001, 20] \times [0.5, 1] \times [0.1, 5]$. Despite the relatively high dimensionality of the search box \mathcal{X} , the computation time for this design problem is equal, on average, to a moderate 15 min 42 s. The genetic/interval algorithm has found the following minimizer: $\mathbf{x}^* = [9.0874 \ -1.2738 \cdot 10^{-2} \ 2.3415 \cdot 10^{-3} \ 7.9450 \ 19.258 \ 0.55621 \ 1.8308]^T$ with nominal index $J(\mathbf{x}^*) = 1.03679$. Also, for the third-order controller the H_∞ constraint is active at two different frequencies. The closed-loop system has three poles that are practically independent of k . Two of them are dominating poles (-0.009163 and -0.004197) while the third one is a fast pole (-19.26). The positions of the other four poles

depending on k are shown in Fig. 13 ($k=0.5$, $k=1.25$, and $k=2$).

4. Conclusions

The problem of synthesizing a fixed-structure robust controller with mixed H_2/H_∞ specifications for an uncertain SISO plant has been faced with the aid of a global optimization technique. Specifically, the H_2/H_∞ synthesis problem can be posed as a semi-infinite optimization problem solvable by means of a genetic/interval algorithm developed by the authors [7,9].

The examples of Section 3 highlight the flexibility and the effectiveness of the proposed approach. In particular, an intrinsic advantage of this approach lies in synthesizing easily implementable low-order controllers that do not require post-design processing for order reduction. A possible limitation of the method could occur in addressing large-scale problems, i.e. problems where the dimension of the design parameter space is high (for example, $l \geq 20$). However, this should not exclude the possibility of successfully addressing more general problems, such as the multi-objective H_2/H_∞ control of multivariable plants.

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Appendix

For the reader's convenience, this appendix exposes succinctly the determinantal method to compute the H_2 norm $\|T(s)\|_2$ of a given strictly stable rational function $T(s)$. This method is due to Katz [13] and was subsequently reported in [21] (see also [16, pp. 8–11]). Defining $J := \|T(s)\|_2^2$ for ease of notation, it is well known that $J = (\int_0^\infty r(t)^2 dt)^{1/2}$, where $r(t)$ is the impulse response of the system whose transfer function is $T(s)$ (cf. [2, p. 96]).

Define $T(s)$ according to

$$T(s) = \frac{B(s)}{A(s)} = \frac{b_1 s^{n-1} + b_2 s^{n-2} + \dots + b_n}{a_0 s^n + a_1 s^{n-1} + \dots + a_n}$$

and assume $a_0 \neq 0$ and $A(s)$ (Hurwitz) stable for which J is well defined. Introduce the polynomial $F(s) := (-1)^{n-1} \{f_1 s^{n-1} + f_2 s^{n-2} + \dots + f_n\}$, $f_i \in \mathbb{R}$, $i = 1, \dots, n$ satisfying the following polynomial identity:

$$B(s)B(-s) = F(-s)A(-s) + F(s)A(s).$$

This relation implies that f_i must satisfy a system of linear equations for which, under the current assumptions, a unique solution always exists; then, by virtue of Parseval's theorem, it follows that

$$J^2 = \frac{f_1}{a_0}.$$

Using Cramer's rule, the following closed-form expression of f_1 can be obtained:

$$f_1 = \frac{\det \begin{pmatrix} g_1 & a_0 & 0 & 0 & \dots & 0 \\ g_2 & a_2 & a_1 & a_0 & \dots & 0 \\ g_3 & a_4 & a_3 & a_2 & \dots & \cdot \\ \cdot & \cdot & \cdot & a_4 & \dots & \cdot \\ \cdot & \cdot & \cdot & \cdot & \dots & \cdot \\ g_n & 0 & \cdot & \cdot & \dots & a_n \end{pmatrix}}{\det \begin{pmatrix} a_1 & a_0 & 0 & 0 & \dots & 0 \\ a_3 & a_2 & a_1 & a_0 & \dots & 0 \\ a_5 & a_4 & a_3 & a_2 & \dots & \cdot \\ \cdot & \cdot & \cdot & a_4 & \dots & \cdot \\ \cdot & \cdot & \cdot & \cdot & \dots & \cdot \\ 0 & 0 & \cdot & \cdot & \dots & a_n \end{pmatrix}}, \quad (27)$$

where

$$g_i = \frac{1}{2} \sum_{j=1}^{2i-1} (-1)^{j-1} b_{2i-j} b_j \quad (b_j = 0 \text{ if } j > n).$$

From (27) we identify the denominator as the n -order Hurwitz determinant associated with $A(s)$. Then,

denoting the numerator of (27) by N and taking into account that $H_n = a_n H_{n-1}$ (where H_{n-1} is the Hurwitz determinant of order $n-1$) we finally obtain

$$J = \left(\frac{N}{a_0 a_n H_{n-1}} \right)^{1/2}. \quad (28)$$