MOTION PLANNING FOR STEERING CAR-LIKE VEHICLES

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Abstract

In this article, using the kinematics model of a car-like vehicle with front-wheel driving we address the path-following problem on the Cartesian plane. Adopting a dynamic inversion approach we determine initial conditions and the steering input in such a way that a front point of the vehicle exactly follows a pre-specified Cartesian path. Motivation for this special motion planning problem arises from the needs of the autonomous vehicle driving using a stereo vision system. Relevant examples, i.e. path-following of lines, circle arcs, and quintic splines, are included with simulations.

1 Introduction

Generally speaking, motion planning for the control of nonholonomic systems concerns finding open-loop control laws to steer the controlled system from a given initial state to a desired final one. Relevant research works on the subject are summarized in the survey of [7] that also reports an extensive bibliography.

In this paper, using a dynamic inversion approach we address the motion planning for steering a car-like vehicle in such a way that a front point of the vehicle exactly follows a pre-specified Cartesian path. Motivation for this special motion planning arises from the needs of the autonomous vehicle driving using a stereo vision system [1].

The general idea behind the chosen approach, i.e. dynamic inversion, is quite simple. Indeed, it prescribes to set forth the desired signals or planning on the output space and then to find the initial conditions and the control input that causes the desired output [3, 6, 9]. It is worth noting that the well-known flatness approach control of nonholonomic systems is in fact a special dynamic inversion performed on the so-called flat output and characterized by a trivial zero-dynamics [4].

Paper's organization. Section 2 exposes the problem formulation and the main results, i.e. Theorem 1 and Theorem 2 that provide the local and global solutions respectively. Particularly relevant is the Corollary 1 that gives a simple sufficient condition for the existence of a global solution in terms of a bound on the curvature along the path to follow. The subsequent Section 3 provides the constructive proofs of the previously exposed results. Section 4 reports three path following examples: a line for which a closed-form steering control is given, a circle arc for which an almost-closed solution is provided, and a quintic spline [8] whose associated solution is gained through numerical integration.

Notation. A curve on the Cartesian $\{x, y\}$-plane be described by a parameterization $\boldsymbol{\gamma}(\lambda) = [\xi(\lambda),\ \eta(\lambda)]^T$ with real parameter $\lambda \in [\lambda_0, \lambda_1]$. The associated "path" is the image of $[\lambda_0, \lambda_1]$ under the vectorial function $\boldsymbol{\gamma}(\lambda)$, i.e. $\boldsymbol{\gamma}([\lambda_0, \lambda_1])$ or more simply Γ. We say that the curve $\boldsymbol{\gamma}(\lambda)$ is regular if there exists $\dot{\boldsymbol{\gamma}}(\lambda)$ over $[\lambda_0, \lambda_1]$ and $\dot{\boldsymbol{\gamma}}(\lambda) \neq 0, \forall \lambda \in [\lambda_0, \lambda_1]$. A curve $\boldsymbol{\gamma}(\lambda)$ is of class C^k if $\boldsymbol{\gamma} \in C^k([\lambda_0, \lambda_1], \mathbb{R}^2)$, i.e. both the coordinate functions $\xi(\lambda)$ and $\eta(\lambda)$ have continuous derivatives up to the kth-order. Denote also with $\kappa(\lambda)$ the scalar curvature associated to every point of the curve $\boldsymbol{\gamma}(\lambda)$ according to the Frenet formulae [5, pag. 109]. The Euclidean norm of a vector $\mathbf{P}$ is denoted with $\|\mathbf{P}\|$.

2 The problem and main results

The motion model of a car-like vehicle with front-wheel driving will be given by the following simplified nonholonomic system:

$$\left\{ \begin{array}{rcl} \dot{x}(t) & = & v \cos\theta(t) \\ \dot{y}(t) & = & v \sin\theta(t) \\ \dot{\theta}(t) & = & \frac{v}{l} \tan\delta(t) \end{array} \right. \qquad (1)$$

with initial conditions $x(0) = x_0$, $y(0) = y_0$, $\theta(0) = \theta_0$. Here (see Fig. 1) x and y are the Cartesian coordinates of the rear axle midpoint $\mathbf{P}$ whose velocity v is constant ($\|\dot{\mathbf{P}}(t)\| = v$ for any t), θ is the vehicle's heading angle, l is the inter-axle distance, and δ, the front wheel angle, is the control variable to steer the vehicle. A distinguished point of the model is $\mathbf{Q}$ — denoted as the "front point" of the vehicle — belonging to

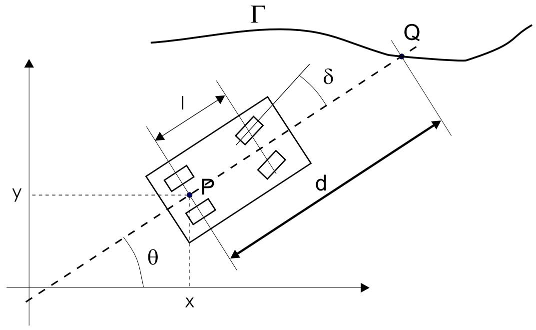


Figure 1: The car-like vehicle with the front point $\mathbf{Q}$.

the vehicle's symmetry axis at a fixed distance d from $\mathbf{P}$, ahead the vehicle. This point could be a physical point of the vehicle or a virtual one belonging to the road scene as viewed in the look-ahead range by the vehicle's vision system. At any time the motion of $\mathbf{Q}$ is described by

$$\mathbf{Q}(t) = \mathbf{P}(t) + d[\cos\theta(t)\ ,\sin\theta(t)]^T\ . \qquad (2)$$

The addressed problem can be introduced as follows.
The problem: *Given a sufficiently smooth Cartesian curve $\boldsymbol{\gamma}$ find a continuous steering control $\delta(t)$ and initial conditions of model (1) in such a way that the motion path of the front point $\mathbf{Q}$ exactly matches the path Γ.*

For the degenerate case of $\mathbf{Q} = \mathbf{P}$, i.e. $d = 0$, the above problem has been solved in [2] by means of an elegant closed-form solution exploiting the curvature function along the curve $\boldsymbol{\gamma}$. For the nondegenerate case $d > 0$ the results below, Theorems 1 and 2, are presented.

Roughly speaking, the philosophy of the following theorems says that if at the initial time $\mathbf{Q}$ belongs to the path Γ (cf. (4)) and the vehicle's direction is not normal to Γ (cf. (5)), then $\mathbf{Q}$ can run over Γ at least for a while. Furthermore if the absolute value of the curvature of Γ is not too large (cf. (8)), for instance less than $1/d$, then $\mathbf{Q}$ will run over the entire Γ. For ease of notation define

$$\mathbf{W}(\theta) := \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}\ . \qquad (3)$$

Theorem 1 (Local solution) *Let be given a curve $\boldsymbol{\gamma} : [0,a] \to \mathbb{R}^2 \quad (0 < a < +\infty)$ of class C^2 with arc-length parameterization, i.e. $\|\dot{\boldsymbol{\gamma}}(\lambda)\| = 1$, $\forall\lambda \in [0,a]$. Assume that the initial state components of model (1) $\mathbf{P}(0) = [x_0\ ,y_0]^T$ and θ_0 satisfy:*

$$\boldsymbol{\gamma}(0) = \mathbf{P}_0 + d\mathbf{W}(\theta_0) \qquad (4)$$

and

$$\dot{\boldsymbol{\gamma}}(0)^T\mathbf{W}(\theta_0) > 0\ . \qquad (5)$$

Then there exist a $t_\epsilon \in \mathbb{R}^+$ and a steering function $\delta \in C^0([0,t_\epsilon],\mathbb{R})$ such that the motion of the front point $\mathbf{Q}$ belongs to Γ, i.e. $\mathbf{Q}(t) \in \Gamma$, $\forall t \in [0,t_\epsilon]$.

Theorem 2 (Global solution) *Let be given a curve $\boldsymbol{\gamma} : [0,a] \to \mathbb{R}^2 \quad (0 < a < +\infty)$ of class C^2 with arc-length parameterization, i.e. $\|\dot{\boldsymbol{\gamma}}(\lambda)\| = 1\ \forall\lambda \in [0,a]$. Assume that the initial state components of model (1) $\mathbf{P}(0) = [x_0\ y_0]^T$ and θ_0 satisfy:*

$$\boldsymbol{\gamma}(0) = \mathbf{P}(0) + d\mathbf{W}(\theta_0) \qquad (6)$$

and

$$\dot{\boldsymbol{\gamma}}(0)^T\mathbf{W}(\theta_0) > 0\ . \qquad (7)$$

Define $\alpha_0 := \theta_0 - \arg\dot{\boldsymbol{\gamma}}(0)$ assume that there exist $\epsilon > 0$ such that $\forall\lambda \in [0,a]$

$$\max\left\{\alpha_0^+ e^{-\frac{2}{\pi d}\lambda} - \int_0^\lambda \kappa^-(\lambda - s)e^{-\frac{2}{\pi d}\lambda}ds,\right.$$
$$\left. -\alpha_0^- e^{-\frac{2}{\pi d}\lambda} + \int_0^\lambda \kappa^+(\lambda - s)e^{-\frac{2}{\pi d}\lambda}ds\right\} \le \frac{\pi}{2} - \epsilon, \qquad (8)$$

where $\alpha_0^- := \min\{\alpha_0, 0\}$, $\alpha_0^+ := \max\{\alpha_0, 0\}$, $\kappa^-(\lambda) := \min\{\kappa(\lambda), 0\}$, and $\kappa^+(\lambda) := \max\{\kappa(\lambda), 0\}$.
Then there exist $t_f \in \mathbb{R}^+$ and a steering function $\delta \in C^0([0,t_f],\mathbb{R})$ such that the motion of the front point $\mathbf{Q}$ exactly matches all Γ, i.e. $\mathbf{Q}([0,t_f]) = \Gamma$.

Corollary 1 *Given a curve $\boldsymbol{\gamma}$ satisfying the assumptions introduced in the above theorems and α_0 defined as in the statement of Theorem 2, if $|\alpha_0| < \pi/2$ and $|\kappa(\lambda)| < 1/d$, $\forall\lambda \in [0,a]$ then there exist $t_f \in \mathbb{R}^+$ and a steering function $\delta \in C^0([0,t_f],\mathbb{R})$ such that the motion of the front point $\mathbf{Q}$ exactly matches all the image Γ of γ. Therefore if γ has infinite length, Γ is entirely covered.*

The proofs of the above results exposed in the next section are fully constructive so that practical applications can be successfully derived (cf. examples of section 4).

3 Proofs

Remark 1 *The control function for the steering angle $\delta(t)$ that appears in equations (1) can be obtained from the function $\theta(t)$ through the relation:*

$$\delta(t) = \arctan(\frac{l}{v}\dot{\theta}(t)) \qquad (9)$$

To be brief, in the following proofs we will determine only function $\theta(t)$. We observe that the continuity order of function $\delta(t)$ is one degree lower than that of function $\theta(t)$. Hence $\theta(t)$ has to be of class C^1 to ensure that $\delta(t)$ is continuous.

The following Lemma 1 shows that the solution to the posed path-following problem can be obtained by solving a particular nonlinear differential equation.

Lemma 1 *Let be given a curve $\boldsymbol{\gamma} : \mathcal{J} \to \mathbb{R}^2,\ \lambda \leadsto \boldsymbol{\gamma}(\lambda) = [\xi(\lambda),\ \eta(\lambda)]^T$ of class C^2 over the real interval $\mathcal{J}$ and with arc-length parameterization, i.e. $\|\dot{\boldsymbol{\gamma}}(\lambda)\| = 1$, $\forall\lambda \in \mathcal{J}$. Let also $\Psi, \Phi : \mathcal{J} \times \mathbb{R} \to \mathbb{R}$ be defined as it follows:*

$$\Phi(\lambda,\tau) = \boldsymbol{\nu}(\lambda)^T\mathbf{W}(\tau), \quad \Psi(\lambda,\tau) = \dot{\boldsymbol{\gamma}}(\lambda)^T\mathbf{W}(\tau), \quad \forall(\lambda,\tau) \in \mathcal{J} \times \mathbb{R} \qquad (10)$$

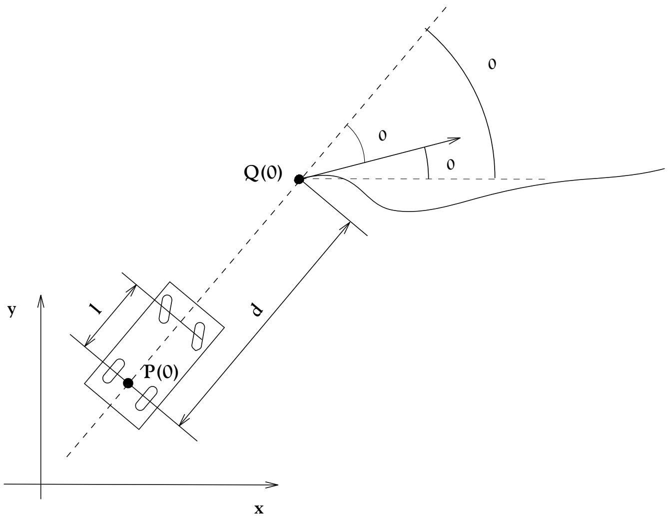


Figure 2: Situation at time $t = 0$.

where $\boldsymbol{\nu}(\lambda) = [\dot{\eta}(\lambda), \ -\dot{\xi}(\lambda)]^T$ is the normal unit vector of $\boldsymbol{\gamma}$ at λ and $\mathbf{W}(\tau)$ is defined according to (3). Let $\mathcal{I}$ be a real interval and assume that there exist two functions $\theta \in \mathcal{C}^1(\mathcal{I}, \mathbb{R})$ and $\mu \in \mathcal{C}^1(\mathcal{I}, \mathcal{J})$ such that

$$\Psi(\mu(t), \theta(t)) \neq 0, \quad \forall t \in \mathcal{I} \tag{11}$$

$$\begin{cases} \dot{\mu}(t) = v\frac{1}{\Psi(\mu(t),\theta(t))} & ,\forall t \in \mathcal{I} \\ \dot{\theta}(t) = \frac{v}{d}\frac{\Phi(\mu(t),\theta(t))}{\Psi(\mu(t),\theta(t))} & ,\forall t \in \mathcal{I} \end{cases} \tag{12}$$

Suppose that at time $t_0 \in \mathcal{I}$, model (1) satisfies the following property:

$$\boldsymbol{\gamma}(\mu(t_0)) = \mathbf{P}(t_0) + d\mathbf{W}(\theta(t_0)) \tag{13}$$

and the control input $\delta(t)$ be given by (9). Then we have

$$\mathbf{Q}(t) = \boldsymbol{\gamma}(\mu(t)) \quad ,\forall t \in \mathcal{I}. \tag{14}$$

Proof. By the hypotheses we made, $\mathbf{Q}, \boldsymbol{\gamma} \circ \mu \in \mathcal{C}^1(\mathcal{I}, \Re^2)$, $\mathbf{Q}(t_0) = \boldsymbol{\gamma} \circ \mu(t_0)$ and $\dot{\mathbf{Q}}(t) = \frac{d(\boldsymbol{\gamma}\circ\mu)}{dt}(t), \quad \forall t \in \mathcal{I}$. Therefore by the fundamental theorem of integral calculus we obtain equation (14). □.

Proof of Theorem 1. By the hypothesis $\Psi(0, \theta_0) = \dot{\boldsymbol{\gamma}}(0)^T\mathbf{W}(\theta_0) > 0$ and being Ψ continuous, there exists $\epsilon > 0$ such that $\Psi(\lambda, \tau) > 0, \ \forall(\lambda, \tau) \in \mathcal{W} := [0, \epsilon] \times [\theta_0 - \epsilon, \theta_0 + \epsilon]$, therefore function $\mathbf{F} : \mathcal{W} \to \mathbb{R}^2, \ (\lambda, \tau) \leadsto (\frac{v}{\Psi(\lambda,\tau)}, -\frac{v}{d}\frac{\Phi(\lambda,\tau)}{\Psi(\lambda,\tau)})$ is a well defined, Lipschitz map. Consequently, by the local existence and uniqueness theorem, there exist $t_\epsilon > 0$ and one and only one pair of functions $\mu \in \mathcal{C}^1([0, t_\epsilon], \mathcal{J})$ and $\theta \in \mathcal{C}^1([0, t_\epsilon], \mathbb{R})$ such that:

$$\begin{cases} (\dot{\mu}(t), \dot{\theta}(t)) = \mathbf{F}(t, \mu(t), \theta(t)) & \forall t \in [0, t_\epsilon] \\ (\mu(0), \theta(0)) = (0, \theta_0) \end{cases} \tag{15}$$

To prove the statement we apply the lemma with $\mathcal{I} = [0, \epsilon], \quad \mathcal{J} = [0, a], \quad t_0 = 0$ and we remark that if we write

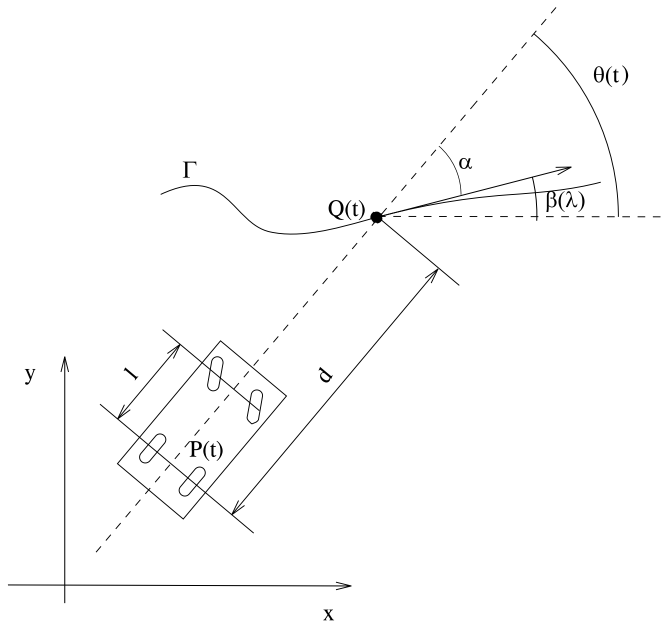


Figure 3: Situation at time $t \neq 0$.

$\mathbf{Q}(t) = \mathbf{P}(t) + d[\cos(\theta(t)), \sin(\theta(t))]^T, \quad \forall t \in \mathcal{I}$ where $\mathbf{P}(t)$ is given by (1), then $\mathbf{Q}(0) = \boldsymbol{\gamma}(0) = \boldsymbol{\gamma}(\mu(0))$ because of equation (4).□

If there exist $t \in \mathbb{R}$ and $\lambda \in [0, a]$ such that $\mathbf{Q}(t) = \boldsymbol{\gamma}(\lambda)$, we call α the angle $\theta(t) - \dot{\gamma}(\lambda)$ between the speed vector of $\mathbf{P}$ and the tangent vector at the point $\boldsymbol{\gamma}(\lambda)$ of Γ(cf. fig. 3), therefore:

$$\dot{\boldsymbol{\gamma}}(\lambda)^T\mathbf{W}(\theta(t)) \neq 0 \iff \alpha \neq \frac{\pi}{2} + K\pi \quad (K \in \mathbb{Z}) \tag{16}$$

Figure 3 illustrates the role of angle α.

Proof of Theorem 2. Let $[0, \sigma[$ the maximum right set of existence of the solution pair (μ, θ) found during the previous proof, satisfying the property

$$\Psi(\mu(t), \theta(t)) > 0 \quad ,\forall t \in [0, \sigma[. \tag{17}$$

We have to prove that, if equation (2) holds, $\mu([0, \sigma[) = [0, a[$. Reasoning by contradiction, we assume that $\overline{\lambda} = \sup \mu([0, \sigma[) < a$. If we show that:

$$\exists c > 0 : \Psi(\mu(t), \theta(t)) \geq c \quad ,\forall t \in [0, \sigma[, \tag{18}$$

then by equations (15), functions μ and θ can be extended to the closed interval $[0, \sigma]$ and $\mu(\sigma) = \overline{\lambda} < a$, being μ a monotone increasing function. Since $\Psi(\overline{\lambda}, \theta(\sigma)) \geq c > 0$, it is sufficient to solve an initial value problem analogous to (15) with $(\overline{\lambda}, \theta(\sigma))$ instead of $(0, \theta_0)$ to contradict the fact that $[0, \sigma]$ is the maximum interval of existence.
Therefore, we show that hypothesis (8) implies property (18). First of all we remark that $\dot{\mu}(t) = \frac{v}{\Psi(\mu(t),\theta(t))} > 0, \quad \forall t \in [0, \sigma[$, hence $\mu : [0, \sigma[\to [0, \overline{\lambda}[$ is monotone increasing and can be inverted. Let $\mu^{-1} : [0, \overline{\lambda}[\to [0, \sigma[$ be its inverse. We

write $\forall\lambda \in [0,\overline{\lambda}[, \quad \beta(\lambda) = \arg(\dot{\boldsymbol{\gamma}}(\lambda))$ and $\kappa(\lambda) = \dot{\beta}(\lambda)$, where $\kappa(\lambda)$ is the curvature of $\boldsymbol{\gamma}$ at λ. We obtain that, $\forall\lambda \in [0,\overline{\lambda}[, \quad \forall\tau \in \mathbb{R}:$

$$\dot{\boldsymbol{\gamma}}(\lambda) = [\cos(\beta(\lambda)), \sin(\beta(\lambda))]^T,$$
$$\boldsymbol{\nu}(\lambda) = [\sin(\beta(\lambda)), -\cos(\beta(\lambda))]^T \ ;$$

and (see equations (10))

$$\Psi(\lambda,\tau) = [\cos(\beta(\lambda)), \sin(\beta(\lambda))]^T[\cos(\tau), \sin(\tau)]$$
$$\Phi(\lambda,\tau) = [\sin(\beta(\lambda)), -\cos(\beta(\lambda))]^T[\cos(\tau), \sin(\tau)];$$

hence:

$$\frac{d}{d\lambda}\Psi(\lambda,\theta(\mu^{-1}(\lambda))) = \frac{1}{d}\Phi^2(\lambda,\theta(\mu^{-1}(\lambda)))+ \\ -\kappa(\lambda)\Phi(\lambda,\theta(\mu^{-1}(\lambda))). \tag{19}$$

Then, if $\forall\lambda \in [0,\overline{\lambda}[, \quad \alpha(\lambda) = \theta(\mu^{-1}(\lambda)) - \beta(\lambda)$ it follows that $\Psi(\lambda,\theta(\mu^{-1}(\lambda))) = \cos(\alpha(\lambda))$ and $\Phi(\lambda,\mu^{-1}(\lambda)) = -\sin(\alpha(\lambda))$ and equation (19) leads to:

$$-\sin(\alpha(\lambda))\dot{\alpha}(\lambda) = \frac{1}{d}\sin^2(\alpha(\lambda)) + \kappa(\lambda)\sin(\alpha(\lambda)), \\ \forall\lambda \in [0,\lambda[\tag{20}$$

Analogously, by deriving $\Phi(\lambda,\theta(\mu^{-1}(\lambda)))$, we obtain, $\forall\lambda \in [0,\lambda[$:

$$-\cos(\alpha(\lambda))\dot{\alpha}(\lambda) = \frac{1}{d}\cos(\alpha)\sin(\alpha(\lambda)) + \kappa(\lambda)\cos(\alpha(\lambda)), \tag{21}$$

Finally equations (20) and (21) imply that $\alpha(\lambda)$ satisfies the following differential equation:

$$\dot{\alpha}(\lambda) = -\frac{1}{d}\sin(\alpha(\lambda)) - \kappa(\lambda) \quad ,\forall\lambda \in [0,\overline{\lambda}[\tag{22}$$

and $\alpha(0) = \theta_0 - \beta(0) = \alpha_0$.

Set $a^+ : [0,\overline{\lambda}[\to \mathbb{R}^+ = \{x \in \mathbb{R} | x \geq 0\}, \ \lambda \rightsquigarrow \alpha^+(\lambda) = \max\{\alpha(\lambda), 0\}$; the function α^+ is absolutely continuous and, by (22), the following differential equation holds almost everywhere in $[0,\overline{\lambda}[$:

$$(\dot{\alpha^+})(\lambda) \leq -\frac{2}{\pi d}\alpha^+(\lambda) - \kappa^-(\lambda), \quad \text{almost everywhere in} [0,\overline{\lambda}[$$

and $\alpha^+(0) = \alpha_0^+$, where $\kappa^+(\lambda) = \max\{\kappa(\lambda), 0\}$. Hence:

$$\alpha^+(\lambda) \leq \alpha_0^+ e^{-\frac{2}{\pi d}\lambda} - \int_0^\lambda \kappa^-(\lambda - s)e^{-\frac{2}{\pi d}s}\, ds, \quad \forall\lambda \in [0,\overline{\lambda}[\tag{23}$$

Likewise we obtain:

$$\alpha^-(\lambda) \geq \alpha_0^- e^{-\frac{2}{\pi d}\lambda} - \int_0^\lambda \kappa^+(\lambda - s)e^{-\frac{2}{\pi d}s}\, ds, \quad \forall\lambda \in [0,\overline{\lambda}[\tag{24}$$

where $\alpha^-(\lambda) = \min\{\alpha(\lambda), 0\}$ and $\kappa^-(\lambda) = \min\{\kappa(\lambda), 0\}$. From equations (23) and (24), it follows that:

$$\alpha_0^- e^{-\frac{2}{\pi d}\lambda} - \int_0^\lambda \kappa^+(\lambda - s)e^{-\frac{2}{\pi d}s}\, ds \leq \alpha^-(\lambda) \leq \alpha(\lambda) \\ \leq \alpha^+(\lambda) \leq \alpha_0^+ e^{-\frac{2}{\pi d}\lambda} - \int_0^\lambda \kappa^-(\lambda - s)e^{-\frac{2}{\pi d}s}\, ds.$$

Therefore if we set

$$c_0 = \max_{\lambda\in[0,\overline{\lambda}[}\left\{\alpha_0^+ e^{-\frac{2}{\pi d}\lambda} - \int_0^\lambda \kappa^-(\lambda - s)e^{-\frac{2}{\pi d}s}\, ds, \right. \\ \left. -\alpha_0^- e^{-\frac{2}{\pi d}\lambda} + \int_0^\lambda \kappa^+(\lambda - s)e^{-\frac{2}{\pi d}s}\, ds\right\}$$

we obtain from equation (8) that $0 \leq c_0 < \frac{\pi}{2}$ and $-c_0 \leq \alpha(\lambda) \leq c_0 \quad ,\forall\lambda \in [0,\overline{\lambda}[$.

In conclusion equation (18) holds with $c = \cos(c_0)$ and this completes the proof. □

Remark 2 *A condition stronger than (8) for Theorem 2 is this*

$$\max\{\alpha_0^+ - \int_0^a \kappa^-(\lambda - s)e^{-\frac{2}{\pi d}s}\, ds, \\ -\alpha_0^- + \int_0^a \kappa^+(\lambda - s)e^{-\frac{2}{\pi d}s}\, ds\} < \frac{\pi}{2}$$

Proof of Corollary 1. If $|\kappa(\lambda)| \leq \frac{1}{d}, \quad \forall\lambda \in [0,a]$ it follows that:

$$\max_{\lambda\in[0,\overline{\lambda}[}\left\{\alpha_0^+ e^{-\frac{2}{\pi d}\lambda} - \int_0^\lambda \kappa^-(\lambda - s)e^{-\frac{2}{\pi d}s}\, ds, \right. \\ \left. -\alpha_0^+ e^{-\frac{2}{\pi d}\lambda} - \int_0^\lambda \kappa^+(\lambda - s)e^{-\frac{2}{\pi d}s}\, ds\right\} \leq \\ \leq \frac{\pi}{2} - e^{-\frac{2}{\pi d}\overline{\lambda}}(\frac{\pi}{2} - |\alpha_0|)$$

The last equation satisfies hypothesis (8) and Theorem 2 yields to the result.□

Remark 3 *It's obvious that if the curve $\boldsymbol{\gamma}$ has an infinite length, his image set can be entirely covered if condition (8) holds in every limited length section of the curve, and this is certainly true if the condition $|\kappa(\lambda)| < \frac{1}{d}$ holds for every $\lambda \in [0, +\infty[$.*

4 Examples

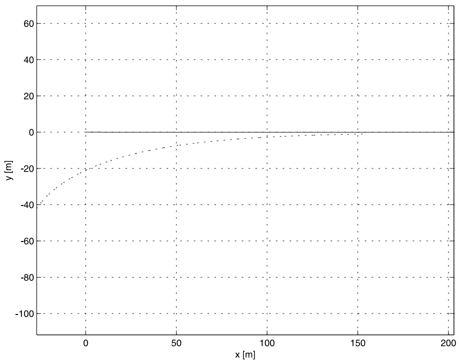
Remark 4 *If condition (16) holds, the solution $\alpha(\lambda)$ of equation (22) allows us to easily find the functions $\mu(t)$ and $\theta(t)$, in fact the first of equations (12) becomes independent from the second one:*

$$\dot{\mu}(t) = v\frac{1}{\cos(\alpha(\mu(t)))} \tag{25}$$

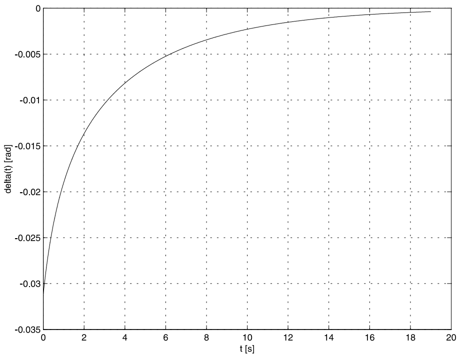
Once found $\mu(t)$ the second of equations (22) assumes the decoupled form:

$$\dot{\theta}(t) = -\frac{v}{d}\tan(\alpha(\mu(\lambda))) \tag{26}$$

This equation allows finding $\theta(t)$. The steering function $\delta(t)$ can then be obtained through (9).



Paths of points $\mathbf{P}(t)$ and $\mathbf{Q}(t)$



Function $\delta(t)$

Figure 4: Straight line

This section describes three path following examples: a straight line, a circle arc in which different cases are considered according to its curvature and a quintic spline whose solution is gained through numerical integration.

1) We consider the case in which the path to be followed is a straight line parallel to the x axis and, initially, the symmetry axis of the vehicle form an angle $\theta_0 = \theta(0)$ with the path itself. In this case $\dot{\boldsymbol{\gamma}}(t) = [1 \;\, 0]^T$. Hence the differential equations (12) become:

$$\begin{cases} \dot{\mu}(t) = v\frac{1}{\cos(\theta)} \\ \dot{\theta}(t) = -\frac{v}{d}\frac{\sin(\theta)}{\cos(\theta)} \end{cases} \tag{27}$$

The solution of the second equation is:

$$\theta(t) = \arcsin(\sin(\theta_0)e^{-\frac{v}{d}t})$$

The first equation can be solved to obtain the law of motion of point $\mathbf{Q}$ over the curve $\boldsymbol{\gamma}$.

In the first graph of figure 4 a continuous line represents the path that point $\mathbf{Q}(t)$ follows and a dashed line represents the path of point $\mathbf{P}(t)$. The second graph represents the steering function $\delta(t)$.

2) We consider a circular path of constant curvature κ (and radius=$\frac{1}{\kappa}$), infinite in the sense that the same shape is outlined over ad over again. Initially the symmetry axis of the vehicle is parallel to the tangent at the point $\mathbf{Q}(0)$ of the curve, hence $\alpha_0 = 0$. In this case we determine the function $\alpha(\lambda)$, that represents the angle between the symmetry axis of the vehicle and the tangent to the curve at the point λ. (See remark 4). In fact this function is easier to find and if condition (16) holds, the existence of control function $\delta(t)$ is guaranteed (see the proof of Theorem 2 for further details).
The function $\alpha(\lambda)$ is given by equation (22), that is

$$\dot{\alpha}(\lambda) = -\frac{1}{d}\sin(\alpha(\lambda)) - \kappa$$

Substituting $x = \tan(\frac{\alpha}{2})$ we obtain:

$$\int_0^{\bar{x}} \frac{dx}{-\frac{2x}{d} - \kappa(1+x^2)} = \int_0^{\bar{\lambda}} d\lambda$$

We distinguish the cases:

a) $\kappa < \frac{1}{d}$

In this case the control function existence and uniqueness for $\lambda \in [0, +\infty[$ is guaranteed by corollary 2. Let x_0, x_1, with $x_0 < x_1$ be the solutions of equation:

$$x^2 + \frac{2x}{d\kappa} + 1 = 0$$

we obtain

$$\alpha(\lambda) = 2 * \arctan(\frac{x_0 x_1(1 - e^{-\lambda(x_1 - x_0)})}{x_1 - x_0 e^{-\lambda(x_1 - x_0)}})$$

Let $\bar{\alpha} = \lim_{\lambda \to +\infty} \alpha(\lambda) = 2\arctan(x_0)$, if $\lambda \to +\infty$ the point $\mathbf{P}$ follows a circle of radius $r_0 = 2 * \sin(\frac{\pi}{4} - \frac{\alpha}{2})$.

Figure 5 represents the trajectory of point $\mathbf{P}$ and function $\delta(t)$ for this case.

b) $\kappa = \frac{1}{d}$

This is a degenerate case in which the hypotheses of corollary 2 are satisfied with the equal sign. We obtain:

$$\alpha = -\arctan(\frac{\lambda\kappa}{\lambda\kappa + 1})$$

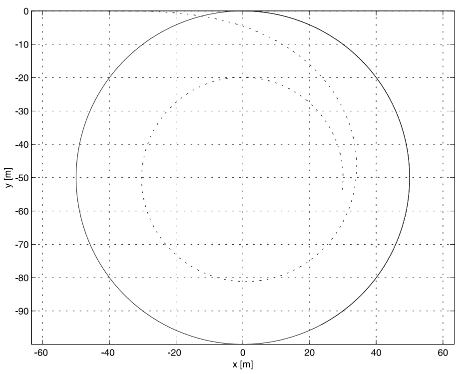
and $\bar{\alpha} = \lim_{\lambda \to +\infty} \alpha(\lambda) = \frac{\pi}{2}$, $r_0 = 0$. Therefore point $\mathbf{P}$ describes a spiral whose radius decreases and reaches 0 as $\lambda \to +\infty$.

c) $\kappa > \frac{1}{d}$

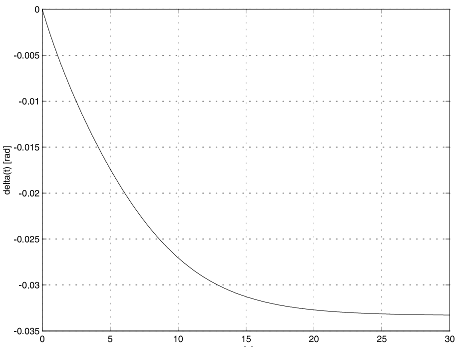
In this case the existence and uniqueness of the control function are guaranteed only in a neighborhood of the initial point. We show that, actually, the function $\alpha(\lambda)$ exists only in an open interval, this means that the given path cannot be followed entirely. We obtain:

$$\alpha(\lambda) =$$
$$= -2\arctan\Big(\frac{\tan(\frac{\lambda\sqrt{\kappa^2 d^2 - 1}}{2d} - \arctan(\frac{1}{\sqrt{\kappa^2 d^2}}))(\sqrt{\kappa^2 d^2 - 1}) + 1}{\kappa d}\Big)$$

This equation has a singularity for $\bar{\lambda} = \frac{\pi}{2}\frac{d}{\sqrt{\kappa^2 d^2 - 1}}$ and the function $\alpha(\lambda)$ exists only in the interval $[0, \bar{\lambda}[$

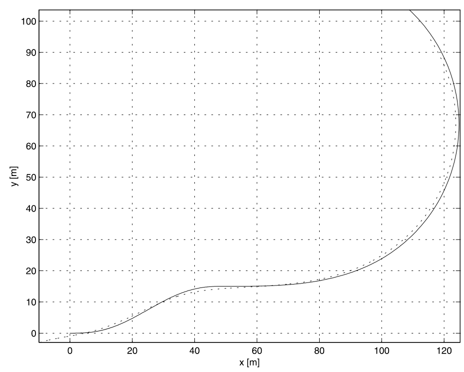


Paths of points $\mathbf{P}(t)$ and $\mathbf{Q}(t)$

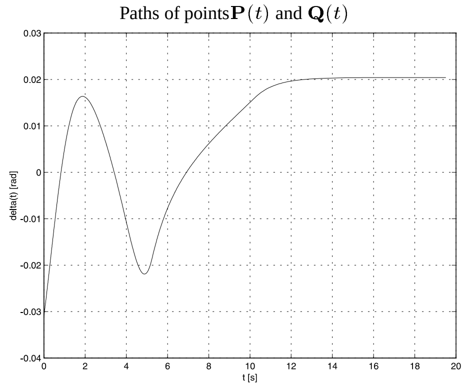


Function $\delta(t)$

Figure 5: Circle



Paths of points $\mathbf{P}(t)$ and $\mathbf{Q}(t)$



Function $\delta(t)$

Figure 6: Quintic spline

3) Although a closed form solution of equations (12) can be found only for very simple trajectories, such as the ones discussed in the first two examples, more complex cases can be solved through numerical integration. For instance, we show in figure 6 the graphs obtained for a quintic spline, a curve obtained splining a composition of fifth degree polynomials[8].

5 Conclusions

In this paper we have proposed a dynamic-inversion-based solution to a path-following problem for car-like vehicles. This open-loop solution could be integrated in a feedback supervisory control for the autonomous driving of a car-like vehicle equipped with a vision system. Future research will be dedicated to this development.

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