Stable input-output inversion of linear dynamical systems

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Introduction

• The general control problem: output tracking (regulation) of dynamical systems

You can approach this problem with a purely feedback strategy (control architecture) but to achieve better performances use a combination of feedforward and feedback action.
Introduction

The standard feedback scheme:

\[ r \rightarrow + \rightarrow C \rightarrow + \rightarrow P \rightarrow y \]

The standard feedforward/feedback scheme:
Two-degrees-of-freedom controller (Horowitz, 1963)

\[ r \rightarrow + \rightarrow C_1 \rightarrow + \rightarrow P \rightarrow y \]

C_1 \rightarrow - \rightarrow C_2 \rightarrow + \rightarrow P \rightarrow y \]
Introduction

- **Input-output inversion** is a radically different approach to design the feedforward action.

Given a dynamical system $\Sigma$:

1. Ask the question: What your output should be? The answer leads to the desired output signal $y_d(t)$.
2. Find the input $u(t)$ that causes $y_d(t)$.

\[ \begin{array}{cc}
? & \sum \\
\rightarrow & \rightarrow \\
& y_d(t)
\end{array} \]
Introduction

When $\Sigma$ is minimum-phase finding $u(t)$ is straightforward: just use any (causal) standard inversion technique.

$\Sigma^{-1}$ is the inverse system of $\Sigma$ (Brockett, Mesarovic 1965)

Silverman’s Inversion Algorithm (1969): for linear multivariable systems.
Introduction

In case $\Sigma$ is nonminimum-phase using the standard inversion leads to an unbounded $u(t)$:

$\Sigma$ is defined by the t.f. $H(s) = \frac{1-s}{s^2 + 7s + 10}$

$\Sigma^{-1}$ is given by $H^{-1}(s) = -s - 8 - \frac{18}{s-1}$

Given $y_d(t)$ the causal input that causes $y_d(t)$ is

$u(t) = \mathcal{L}^{-1} \left[ H^{-1}(s)Y_d(s) \right]$

$u(t) = -Dy_d(t) - 8y_d(t) - 18 \int_0^t e^{t-\tau} y_d(\tau) d\tau$

the integral is unbounded...
Introduction

• Finding a (noncausal) bounded $u(t)$ is possible (stable input-output inversion): Bayo 1987; Devasia, Chen, Paden 1996; Hunt, Meyer, Su 1996.
• They deduced solutions using state-space models of $\Sigma$ for linear and nonlinear systems.
• We propose a transfer function approach (Pallastrelli, Piazzi 2005) to derive the bounded $u(t)$ in the context of linear, continuous-time, scalar systems.
The stable input-output inversion problem

- Consider a linear, scalar system $\Sigma$ described by the transfer function

$$H(s) = k \frac{s^m + b_{m-1}s^{m-1} + \cdots + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_0}$$

$H(s)$ is stable, nonminimum-phase without j-axis zeros.

- The set of all input-output pairs is defined by the behavior

$$\mathbf{B} := \left\{ (u(t), y(t)) \in P^m \times P^n : \text{weak solution of } \sum_{i=0}^{n} a_i \dot{y} = k \sum_{i=0}^{m} b_i \dot{u} \right\}$$

$P^n \triangleq$ set of all piecewise $C^n$-continuous functions defined over $(-\infty, +\infty) \equiv \mathbb{R}$
The stable input-output inversion problem

- **Proposition** [Poldermann, Willems 1998]
  Consider any pair \((u(t), y(t)) \in B\). Then given \(l\), a nonnegative integer,
  \[ u(t) \in C^l \quad \text{if and only if} \quad y(t) \in C^{\rho+l}. \]
  [\(\rho\) is the relative degree of \(\Sigma\)]

- **The SIOI Problem:**
  Given a desired bounded output \(y_d(t) \in BC^\rho\) with \(y_d(t) = 0\) for \(t < 0\)
  find a bounded input \(u_d(t) \in BC^0\) such that
  \[ \left( u_d(t), y_d(t) \right) \in B \]
The structure of the (unstable) standard inverse

- The **standard inverse** function is

  \[ u(t) = \mathcal{L}^{-1}\left[ H^{-1}(s)Y_d(s) \right] \]

  \[ u(t) = 0 \quad \text{for} \quad t < 0 \quad \text{and} \quad (u(t), y_d(t)) \in \mathbf{B} \]

  but \( u(t) \) is unbounded, i.e. \( u(t) \notin BC^0 \)

\[
H^{-1}(s) = \xi_\rho s^\rho + \xi_{\rho-1}s^{\rho-1} + \ldots + \xi_0 + H_0(s)
\]

\[
H_0(s) = \frac{d(s)}{b^-(s)} + \frac{e(s)}{b^+(s)}
\]

\( b^-(s) \) and \( b^+(s) \) are monic polynomials: their roots are the system zeros with negative and positive real parts respectively.
The structure of the (unstable) standard inverse

- \( \eta^-(t) \) and \( \eta^+(t) \) be the analytical extentions of \( \mathcal{L}^{-1}\left[\frac{d(s)}{b^-(s)}\right] \)

and \( \mathcal{L}^{-1}\left[\frac{e(s)}{b^+(s)}\right] \) over the space of the Bohl functions respectively.

- Then for all \( t \in \mathbb{R} \)

\[
u(t) = \xi_{\rho} D^\rho y_d(t) + \xi_{\rho-1} D^{\rho-1} y_d(t) + \ldots + \xi_0 y_d(t)
\]

\[
+ \int_0^t \eta^{-}(t - \tau) y_d(\tau) d\tau + \int_0^t \eta^{+}(t - \tau) y_d(\tau) d\tau
\]
The structure of the (unstable) standard inverse

Proposition 1:
There exist coefficients $d_i \in \mathbb{R}$ such that for all $t \in \mathbb{R}$:

$$
\int_0^t \eta^+(t-\tau)y_d(\tau)d\tau = \sum_{i=1}^{m^+} d_i m_i^+(t) - \int_0^{+\infty} \eta^+(t-\tau)y_d(\tau)d\tau
$$

$m_i^+(t)$ $i = 1, \ldots, m^+$ are the function modes of the unstable zero dynamics.

If $z$ is a real zero with multiplicity $h$ then the associated modes are:

$$\left\{ e^{zt}, te^{zt}, t^2 e^{zt}, \ldots, t^h e^{zt} \right\}$$

If $\sigma \pm \omega$ are complex zeros with multiplicity $h$ then the associated modes:

$$\left\{ e^{\sigma t} \sin \omega t, e^{\sigma t} \cos \omega t, te^{\sigma t} \sin \omega t, te^{\sigma t} \cos \omega t, \ldots, t^{h-1} e^{\sigma t} \sin \omega t, t^{h-1} e^{\sigma t} \cos \omega t \right\}$$
The structure of the (unstable) standard inverse

Proof (constructive): For simplicity, we consider the zeros real and simple (the general case is reported in Pallastrelli, Piazzi 2005 IFAC World Congress)

\[
\eta^+(t) = \sum_{i=1}^{m^+} r_i e^{z_i t} \\
d_i = r_i \mathcal{L}[y_d(t)]_{s=z_i} = r_i Y_d(z_i), \quad i = 1, 2, \ldots, m^+
\]

\[
\int_0^t \eta^+(t-\tau)y_d(\tau)d\tau = \int_0^t \left( \sum_{i=1}^{m^+} r_i e^{z_i(t-\tau)} \right) y_d(\tau)d\tau = \sum_{i=1}^{m^+} \left( \int_0^t r_i e^{z_i(t-\tau)} y_d(\tau)d\tau \right)
\]

\[
= \sum_{i=1}^{m^+} \left( -\int_0^t r_i e^{z_i(t-\tau)} y_d(\tau)d\tau + \int_0^{+\infty} r_i e^{z_i(t-\tau)} y_d(\tau)d\tau - \int_0^{+\infty} r_i e^{z_i(t-\tau)} y_d(\tau)d\tau \right)
\]

\[
= \sum_{i=1}^{m^+} \left( r_i \int_0^{+\infty} e^{-z_i \tau} y_d(\tau)d\tau \right) e^{z_i t} - \int_0^{+\infty} r_i e^{z_i(t-\tau)} y_d(\tau)d\tau
\]

\[
= \sum_{i=1}^{m^+} d_ie^{z_i t} - \sum_{i=1}^{m^+} \int_t^{+\infty} r_i e^{z_i(t-\tau)} y_d(\tau)d\tau = \sum_{i=1}^{m^+} d_i m_i^+(t) - \int_t^{+\infty} \eta^+(t-\tau)y_d(\tau)d\tau
\]
The structure of the (unstable) standard inverse

**Proposition 2:**

\[ \int_{t}^{+\infty} \eta^{+}(t - \tau) y_{d}(\tau) d\tau \] is bounded over \( \mathbb{R} \).

**Proof (a sketch of)**

\[ \forall t \in \mathbb{R} \quad \exists K, z > 0 \quad \exists \quad \left| \eta^{+}(t - \tau) \right| \leq Ke^{-z(t-\tau)} \quad \forall \tau \geq t \]

\[ \left| \int_{t}^{+\infty} \eta^{+}(t - \tau) y_{d}(\tau) d\tau \right| \leq \int_{t}^{+\infty} \left| \eta^{+}(t - \tau) \right| \left| y_{d}(\tau) \right| d\tau \]

\[ \leq \int_{t}^{+\infty} Ke^{-z(t-\tau)} Bd\tau = \frac{KB}{z} \quad \Box \]
The structure of the (unstable) standard inverse

Summarizing: for any $t \in \mathbb{R}$

$$u(t) = \xi_\rho D^\rho y_d(t) + \xi_{\rho - 1} D^{\rho - 1} y_d(t) + \ldots + \xi_0 y_d(t)$$

$$+ \int_0^t \eta^-(t - \tau) y_d(\tau) d\tau - \int_t^{+\infty} \eta^+(t - \tau) y_d(\tau) d\tau + \sum_{i=1}^{m^+} d_i m_i^+(t)$$

The $d_i$'s can be determined via closed-form expressions.
A new formula for the stable solution

Property

\[
\left( \sum_{i=1}^{m^-} \mu_i m_i^-(t) + \sum_{j=1}^{m^+} \nu_j m_j^+(t) , 0 \right) \in \mathcal{B} \quad \forall \mu_i, \nu_j \in \mathbb{R}
\]

Proof (hint): Consider the differential equation of \( \Sigma \) and factorize the differential operator acting on the input…

• **System interpretation:** Injecting at the input the unstable zero modes does not affect the output.

Let \( z > 0 \) be a zero of \( \Sigma \). Hence \( (\varepsilon e^{zt}, 0) \in \mathcal{B} \).

Choose a small \( \varepsilon > 0 \) then \( \varepsilon e^{zt} \approx \varepsilon e^{zt} 1(t) \).
A new formula for the stable solution

The property of the behaviour set implies that:

$$\left( - \sum_{i=1}^{m^+} d_i m^+_i (t) , 0 \right) \in \mathcal{B}$$

On the other hand, the unstable input $u(t)$ causes the desired bounded $y_d(t)$:

$$\left( u(t) , y_d (t) \right) \in \mathcal{B}$$

By linear superposition we obtain

$$\left( u(t) - \sum_{i=1}^{m^+} d_i m^+_i (t) , y_d (t) \right) \in \mathcal{B}$$

This is the sought stable solution (by virtue of exact mathematical cancellation)
A new formula for the stable solution

The solution of the SIOI problem is given by

\[
U_d(t) = \xi_{\rho} D^\rho y_d(t) + \xi_{\rho-1} D^{\rho-1} y_d(t) + \ldots + \xi_0 y_d(t) \\
+ \int_0^t \eta^- (t - \tau) y_d(\tau) d\tau - \int_{t}^{+\infty} \eta^+ (t - \tau) y_d(\tau) d\tau
\]

\[U_d(t) \text{ is bounded over } \mathbb{R} \text{ and } (U_d(t), Y_d(t)) \in B.\]

This closed-form expression can be exploited by numerical as well as symbolic procedures.
A new formula for the stable solution

- When $t < 0$, $u_d(t) = -\sum_{i=1}^{m^+} d_i m_i^+(t)$
  hence $u_d(t)$ is a noncausal signal.

- Define $\eta(t) := 1(t)\eta^-(t) - 1(-t)\eta^+(t)$
  and rewrite the stable solution as

$$u_d(t) = \sum_{i=0}^{\rho} \xi_i D^i y_d(t) + \int_{-\infty}^{+\infty} \eta(t-\tau)y_d(\tau)d\tau$$

System interpretation: $\int_{-\infty}^{+\infty} \eta(t-\tau)y_d(\tau)d\tau$ is the noncausal bounded output of the zero dynamics driven by the signal $y_d(t)$. 

A. Piazzi

Stable Input-Output Inversion
A new formula for the stable solution

- Applying the SIOI solution: feedforward regulation of nonminimum-phase system

As an example consider the system \( H(s) = -\frac{(s-1)(s+1)}{(s+2)(s^2 + s + 2)} \)

The feedforward regulation problem is about designing a control input to obtain an output transition.

**First step:** Choose the desired output transition \( y_d(t) \). A convenient choice may be the *transition polynomials* (Piazzi, Visioli, Automatica 2001).
A new formula for the stable solution

\[
y_d(t, \tau) = \begin{cases} 
0 & \text{if } t \leq 0 \\
\frac{(2k+1)!}{k!\tau^{2k+1}} \sum_{i=0}^{k} \frac{(-1)^{k-i}}{(k-i)!i!(2k+1-i)} \tau^i t^{2k+1-i} & \text{if } 0 \leq t \leq \tau \\
1 & \text{if } t \geq \tau
\end{cases}
\]

\(y_d(t; \tau) \in C^k(\mathbb{R})\); They were deduced by solving the polynomial interpolation problem \(D^i y_d(0) = 0, \ i = 0, 1, \ldots, k; \ y_d(\tau) = 1, \ D^i y_d(\tau) = 0, \ i = 1, 2, \ldots, k.\)

If \(k = 1\) then for \(t \in [0, \tau]\):

\[
y_d(t; \tau) = -2 \frac{t^3}{\tau^3} + 3 \frac{t^2}{\tau^2}
\]
A new formula for the stable solution

Second step: Perform the stable input-output inversion.

\[ H^{-1}(s) = -\frac{1}{4} s - 1 - \frac{1}{4} \cdot \frac{7s + 8}{(s - 1)(s + 1)} \]

\[ = -\frac{1}{4} s - 1 + \frac{1/8}{s + 1} - \frac{15/8}{s - 1} \]

\[ \eta^-(t) = \frac{1}{8} e^{-t}, \quad \eta^+(t) = -\frac{15}{8} e^t \]

\[ u_d(t) = -\frac{1}{4} Dy_d(t; \tau) - Dy_d(t; \tau) + \int_0^t \frac{1}{8} e^{-(t-v)} y_d(v; \tau) dv \]

\[ + \int_t^{+\infty} \frac{15}{8} e^{t-v} y_d(v; \tau) dv \]

A. Piazzi

Stable Input-Output Inversion
A new formula for the stable solution

\[ \tau = 0.94 \text{ s} \]
An automatic inversion scheme

- It is possible to devise an automatic inversion scheme for real-time implementation provided that $y_d(t)$ is known in advance of preview time $\Delta T$.

The idea is to use $\int_{t}^{+\infty} \eta^+(t-\tau)y_d(\tau)d\tau \approx \int_{t}^{t+\Delta T} \eta^+(t-\tau)y_d(\tau)d\tau$

and then compute the definite integral with an approximation rule.

Using a $n$-point Cavalieri-Simpson formula with $\tau_i := i\frac{\Delta T}{n}$ $i = 1, \cdots, n$ the input $u_d(t)$ can be approximate by

$$\tilde{u}(t) := \sum_{i=0}^{\rho} \xi_i D^i y_d(t) + \eta^-(t) \ast y_d(t)$$

$$- \frac{\Delta T}{6n} \left\{ \eta^+(0)y_d(t) + \eta^+(-\tau_n)y_d(t+\tau_n) \right.$$  

$$+ 2 \sum_{i=1}^{n-1} \eta^+(-\tau_i)y_d(t+\tau_i)$$

$$+ 4 \sum_{i=1}^{n} \eta^+\left(\frac{\Delta T}{2n} - \tau_i\right)y_d\left(t - \frac{\Delta T}{2n} + \tau_i\right) \right\}$$
An automatic inversion scheme
An example

- *Feedforward end-point control of a flexible link* [Piazzi, Visioli 2001]. The transfer function between the hub angular position and the link end-point is nonminimum-phase:

\[
H(s) = -0.16 \frac{(s - 9.31)(s + 6.93)}{(s + 1.16)^2 + 2.99^2}
\]

1. Output planning. Choose the desired output to obtain a smooth transition from 0 rad to 0.1 rad: a transition polynomial over the time interval [0, 0.8].

\[
y_d(t) = \begin{cases} 
0 & \text{if } t < 0 \\
0.1 \left( 10 \frac{t^3}{0.8^3} - 15 \frac{t^4}{0.8^4} + 6 \frac{t^5}{0.8^5} \right) & \text{if } t \in (0, 0.8) \\
0.1 & \text{if } t > 0.8 
\end{cases}
\]

Note that \( y_d(t) \in BC^2 \).
2. **Stable inversion.** Determine the bounded noncausal input $u_d(t)$ by using: (a) the new inversion formula (analytic method); (b) the automatic inversion scheme.

(a) **Analytic method**

$$u_d(t) = \begin{cases} 
0.0291e^{9.31t} & \text{if } t < 0 \\
0.204e^{-6.93t} - 0.0000838e^{9.31t} & \\
-0.175 + 1.687t - 3.63t^2 + 3.05t^3 - 1.93t^4 + 1.82t^5 & \text{if } t \in [0, 0.8] \\
0.100 - 6.302e^{-6.93t} & \text{if } t > 0.8
\end{cases}$$

(b) **Automatic inversion scheme** with preview time $\Delta T = 1$ sec and $n = 6, 8, 10$. 
An example

- The noncausal input $u_d(t)$ obtained through stable inversion

When $u_d(t)$ is applied to system $\Sigma$ it determines the so-called preaction control.
An example

• Simulation results

The automatic inversion scheme can be effective even with $n = 8$ (16 delay blocks).

A. Piazzi  Stable Input-Output Inversion
On feedforward/feedback strategies

- Any feedforward technique (when used alone) has the drawback to be not robust against modeling errors, perturbations, and disturbances.
- So, inversion-based feedforward control can achieve high performances but it is fragile (nevertheless it has been successfully proposed for a variety of practical applications…)
- When the aim is to achieve as much as possible robust high performances, the issue is then on how to effectively combine a feedforward action with a feedback one: two basic strategies are reported in the technical literature.
On feedforward/feedback strategies

Devasia Scheme (June 2000 ACC, 2002 TAC) based on the earlier Devasia-Chen-Paden scheme 1996 TAC

In computing $y_d - y$ the preview time must be taken into account.

The stable inversion depends on the controlled plant only.
On feedforward/feedback strategies

Piazzi-Visioli Scheme (December 2000 CDC, 2001 TAC)

The stable inversion depends on both the controlled plant and the feedback controller C.
On feedforward/feedback strategies

• The PV scheme can be regarded as the generalization of the classic two-degrees-of-freedom controller scheme.

• The PV scheme is compatible with the Internal Model Principle (Francis, Wohnham 1976).

• The D scheme is simpler to implement than the PV one.

• Two design approaches can be pursued with the PV scheme:
  1. Design the controller in a standard way and then use the stable inversion to improve the performances.
  2. Design simultaneously the feedback controller and the stable inversion.
On feedforward/feedback strategies

• The D scheme has been proposed for aircraft control, hard-disk control, piezo-based nano-positioner, etc.

• The PV scheme has been proposed for the regulation of electro-mechanical devices and process control application.

Many open research problems remain on feedforward/feedback strategies (comparisons, improvements, robustness, design, optimality).
References